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Algebra Lect 13
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Euclidean domains are PIDs

A Euclidean domain is an integral domain $\mathbb{A}$ with a function $\mathrm{size}: \mathbb{A}-\{0\} \to \mathbb{Z}_{\geq 0}$ such that

if $a, b \in \mathbb{A}$ and $a \neq 0$ then there exist $q, r \in \mathbb{A}$ such that $b = aq + r$ and either
$r = 0$ or $\mathrm{size}(r) < \mathrm{size}(a)$

Proposition If $\mathbb{A}$ is a Euclidean domain then $\mathbb{A}$ is a PID.

Proof:

To show: If $\mathcal{J}$ is an ideal of $\mathbb{A}$ then there exists $m \in \mathbb{A}$ such that $\mathcal{J} = m\mathbb{A}$.

Assume $\mathcal{J}$ is an ideal of $\mathbb{A}$.

Let $m \in \mathcal{J}$ such that

$$\mathrm{size}(m) = \inf\{\mathrm{size}(j) \mid j \in \mathcal{J}\}.$$

To show: $\mathcal{J} = m\mathbb{A}$

To show: (a) $m\mathbb{A} \subseteq \mathcal{J}$

(b) $\mathcal{J} \subseteq m\mathbb{A}$

(a) Since $m \in \mathcal{J}$ and $\mathcal{J}$ is an ideal then if $a \in \mathbb{A}$ then $am \in \mathcal{J}$.

So $m\mathbb{A} \subseteq \mathcal{J}$.

(b) To show: If $j \in J$ then $j \in mA$.

Assume $j \in J$.

Case 1: $j=0$. Then $j = 0 \cdot m \in mA$.

Case 2: $j \neq 0$. Then there exist $q, r \in A$ such that

$$j = mq + r \text{ and either } r = 0 \text{ or } \mathrm{size}(r) < \mathrm{size}(m).$$

So $r = j - mq \in J$.

Since $\mathrm{size}(r)$ is not less than $\mathrm{size}(m)$ then $r = 0$.

So $j = mq$ and $j \in mA$.

So $J \subseteq mA$.

So $J = mA$.

So A is a PID //.

Example Let $A = \mathbb{Z}[x]$ with

$$\begin{aligned} \mathrm{size}: A &\to \mathbb{Z}_{\geq 0} \\ p &\mapsto \deg(p) \\ c_0 + \cdots + c_\ell x^\ell &\mapsto \ell \quad \text{if } c_\ell \neq 0. \end{aligned}$$

Let $J = A\text{span}\{2, x\}$

$$= \{ 2 \cdot f(x) + x g(x) \mid f(x), g(x) \in \mathbb{Z}[x] \}.$$

Let $m=2$.

Let $j=x$. Then find $q,r \in \mathbb{Z}[x]$ such that

$$x = 2\cdot q + r \quad \text{with } r=0 \text{ or } \deg(r)<\deg(x).$$

If $R=\mathbb{Q}[x]$ then

$$x = 2\left(\tfrac{1}{2}x\right)+0, \text{ but } \tfrac{1}{2}x \notin \mathbb{Z}[x].$$

In fact, $\mathbb{Z}[x]$ is not a PID since

$J=\mathbb{A}\text{-span}\{2,x\}$ is not a principal ideal.

<u>A PID satisfies ACC</u>

<u>Proof</u> To show: If $0=I_0 \subseteq I_1 \subseteq \cdots \subseteq \mathbb{A}$ is a chain of ideals then there exists $k\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{>k}$ then $I_n=I_k$.

Assume $0=I_0\subseteq \cdots \subseteq \mathbb{A}$ is a chain of ideals in $\mathbb{A}$.

To show: There exists $k\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{>k}$ then $I_n=I_k$.

Let

$$I_{un} = \bigcup_{j\in\mathbb{Z}_{>0}} I_j.$$

Then I_{un} is an ideal of $\mathbb{A}$.

Since $\mathbb{A}$ is a PID then there exists $d \in \mathbb{A}$ such that $I_{un} = d\mathbb{A}$.

Let $k \in \mathbb{Z}_{>0}$ be such that $d \in I_k$.

To show: If $n \in \mathbb{Z}_{>k}$ then $I_n = I_k$.

Assume $n \in \mathbb{Z}_{>k}$.

Then $I_k \subseteq I_n \subseteq I_{un} = d\mathbb{A} \subseteq I_k$.

So $I_k = I_n$.

So $\mathbb{A}$ satisfies ACC. //.

<u>Proposition</u> Let R be a commutative ring. Then R satisfies

(Cancellation law) If $a, b, c \in R$ and $c \neq 0$ and $ac = bc$ then $a = b$

if and only if R satisfies

(No zero divisors) If $a, b \in R$ and $ab = 0$ then either $a = 0$ or $b = 0$.

<u>Proof</u> $\Rightarrow$: Assume R satisfies the cancellation law.

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To show: If $a, b \in R$ and $ab = 0$ then $a = 0$ or $b = 0$.

Assume $a, b \in R$ and $ab = 0$.

To show: $a = 0$ or $b = 0$.

Assume $a \neq 0$

To show: $b = 0$.

Since $ab = 0 = a \cdot 0$ ~~and a = then~~ and $a \neq 0$ the cancellation law gives $b = 0$.

$\Leftarrow$ Assume R satisfies no zero divisors.

To show: If $a, b, c \in R$ and $c \neq 0$ and $ac = bc$ then $a = b$.

Assume $a, b, c \in R$ and $c \neq 0$ and $ac = bc$.

Then $ac - bc = (a-b)c = 0$.

Since $c \neq 0$ then no zero divisors gives $a - b = 0$.

So $a = b$. //.