Algebra Lect. 12
A. Ram

A <u>unique factorization domain</u> or <u>UFD</u>, is a commutative ring A such that

(a) (Cancellation Law) If $a, b, c \in A$ and $c \neq 0$ and $ac = bc$ then $a = b$.

(b) (Existence of factorizations) If $x \in A$ then there exist $n \in \mathbb{Z}_{>0}$ and irreducible $p_1, \ldots, p_n \in A$ such that $x = p_1 p_2 \cdots p_n$.

(c) (Uniqueness of factorizations). If $x \in A$ and $n, m \in \mathbb{Z}_{>0}$ and $p_1, \ldots, p_n, q_1, \ldots, q_m \in A$ are irreducible and $u \in A^\times$ and

$$x = p_1 \cdots p_n \quad \text{and} \quad x = u q_1 \cdots q_m$$

then $m = n$ and there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $u_1, \ldots, u_n \in A^\times$ such that

$$\text{if } i \in \{1, \ldots, n\} \text{ then } p_i = u_i q_{\sigma(i)}$$

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A <u>principal ideal domain</u>, or <u>PID</u>, is a commutative ring $\mathbb{A}$ such that

(a) (Cancellation law) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

(b) (Ideals are principal) If $I \subseteq \mathbb{A}$ is an ideal of $\mathbb{A}$ then there exists $d \in \mathbb{A}$ such that $I = d\mathbb{A}$,

where $d\mathbb{A} = \{ad \mid a \in \mathbb{A}\} = \mathbb{A}\text{-span}\{d\}$.

An <u>integral domain</u> is a commutative ring $\mathbb{A}$ such that

(Cancellation law) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

<u>Theorem</u> If $\mathbb{A}$ is a PID then $\mathbb{A}$ is a UFD.

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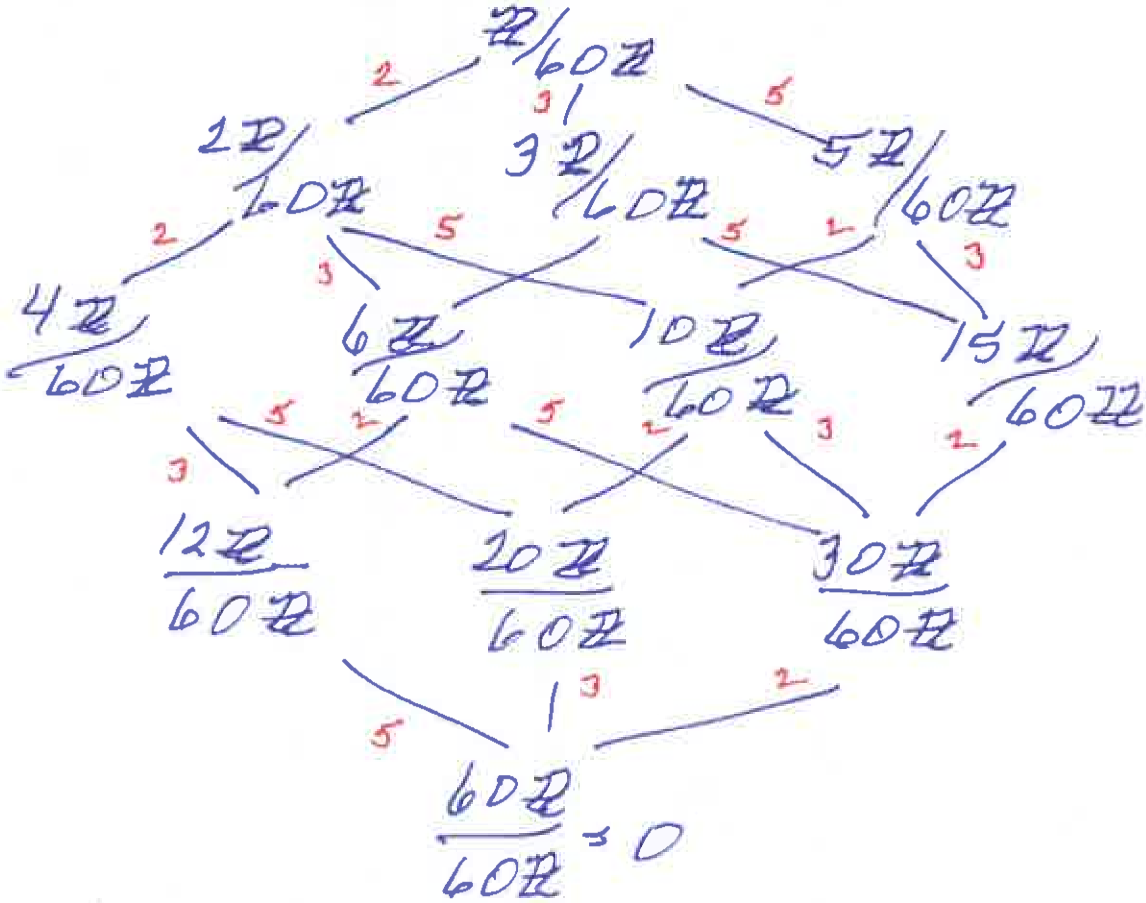
Composition series

Let R be a ring and let M be an R-module. The lattice of submodules of M is

$$\mathcal{S}_0^M = \left\{ \begin{array}{l} R\text{-modules } N \\ \text{with } 0 \subseteq N \subseteq M \end{array} \right\}$$

partially ordered by inclusion.

Example $R = \mathbb{Z}$ and $M = \mathbb{Z}/60\mathbb{Z}$



Factorizations of 60 are "the same" as maximal chains from 0 to $\mathbb{Z}/60\mathbb{Z}$ in $\mathcal{S}_0^{\mathbb{Z}/60\mathbb{Z}}$.

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Let N be a submodule of M. The <u>lattice of submodules between N and M</u> is

$$S_N^M = \left\{ \begin{array}{l} \text{submodules } P \\ \text{with } N \subseteq P \subseteq M \end{array} \right\}$$

partially ordered by inclusion.

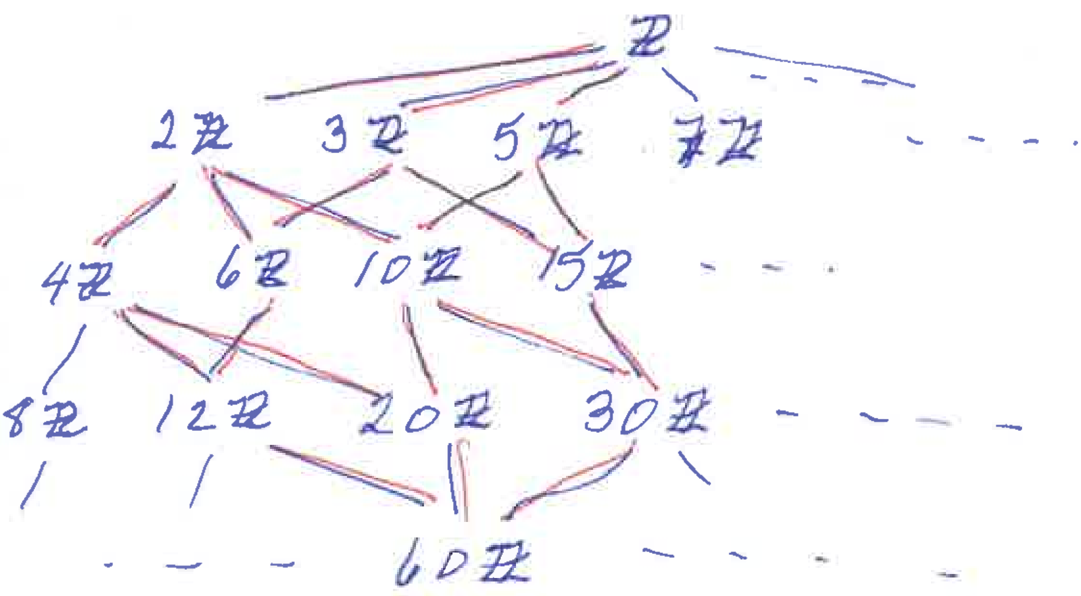
<u>Correspondence theorem</u> The map

$$\begin{array}{ccc} S_N^M & \longrightarrow & S_0^{M/N} \\ P & \longmapsto & P/N = \{p+N \mid p \in P\} \\ \{p \in M \mid p+N \in \Gamma\} & \longleftarrow\!\shortmid & \Gamma \end{array}$$

is an isomorphism of posets.

<u>Proof</u>: Apply proof machine.

<u>Example</u> $R = \mathbb{Z}$ and $M = \mathbb{Z}$



$S_{60\mathbb{Z}}^{\mathbb{Z}} \cong S_0^{\mathbb{Z}/60\mathbb{Z}}$

Let M be an R-module.

The R-module M satisfies the <u>ascending chain condition</u>, or <u>ACC</u>, if

increasing sequences in $\mathcal{S}_0^M$ are finite.

The R-module M satisfies the <u>descending chain condition</u>, or <u>DCC</u>, if

decreasing sequences in $\mathcal{S}_0^M$ are finite.

The R-module M is <u>simple</u> if

$$\mathcal{S}_0^M = \{0, M\} \qquad \begin{matrix} M \\ | \\ 0 \end{matrix}$$

A <u>finite composition series of M</u> is a chain in $\mathcal{S}_0^M$,

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \quad \text{with } n \in \mathbb{Z}_{>0}$$

and if $i \in \{1, \ldots, n\}$ then M_i / M_{i-1} is simple.

Jordan-Hölder Theorem

Let R be a ring and let M be an R-module.

(a) (Existence of composition series)
If M satisfies ACC and DCC then M has a finite composition series.

(b) (Uniqueness of composition series)
Any two composition series for M,

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \quad \text{and}$$

$$0 = M_0' \subseteq M_1' \subseteq \cdots \subseteq M_m' = M$$

have the same length $(m = n)$ and the same composition factors i.e., there exists a permutation $\sigma: \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that if $i \in \{1, \ldots, n\}$ then

$$\frac{M_i}{M_{i-1}} \cong \frac{M'_{\sigma(i)}}{M'_{\sigma(i)-1}}.$$