1.13 Lecture 12: Jordan normal form

Let $\mathbb{F}$ be a field and let $p(x) = x^r - a_{r-1}x^{r-1} - \cdots - a_1x - a_0 \in \mathbb{F}[x]$.

Rational canonical form. The matrix of the action of x on

$$\frac{\mathbb{F}[x]}{p(x)\mathbb{F}[x]} \quad \text{with respect to the } \mathbb{F}\text{-basis} \qquad \{x^{r-1}, x^{r-2}, \ldots, x, 1\}$$

is the $r \times r$ matrix

$$J_1(p(x)) = \begin{pmatrix} a_{r-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{r-2} & 0 & 1 & 0 & \cdots & 0 & 0 \\ a_{r-3} & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \\ a_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ a_0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{r\times r}(\mathbb{F}).$$

The d-Jordan block for $p(x) = x - \lambda$. Let $\lambda \in \mathbb{F}$ and $d \in \mathbb{Z}_{>0}$. The matrix of the action of x on

$$\frac{\mathbb{F}[x]}{(x-\lambda)^d\mathbb{F}[x]} \quad \text{with respect to the } \mathbb{F}\text{-basis} \qquad \{1, (x-\lambda), (x-\lambda)^2, , \ldots, (x-\lambda)^{d-1}\}$$

is

$$J_d(x-\lambda) = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \lambda \end{pmatrix} \in M_{d\times d}(\mathbb{F}),$$

since $x(x-\lambda)^j = (\lambda + x - \lambda)(x-\lambda)^j = \lambda(x-\lambda)^j + (x-\lambda)^{j+1}$.

The d-Jordan block for $p(x) = x^2 - bx - c$. Let $d \in \mathbb{Z}_{>0}$ and let

$$p(x) = x^2 - bx - c \in \mathbb{F}[x].$$

The matrix of the action of x on $\frac{\mathbb{F}[x]}{p(x)^d\mathbb{F}[x]}$ with respect to the basis

$$\{x, 1, p(x)x, p(x), p(x)^2x, p(x)^2, \ldots, p(x)^{d-1}x, p(x)^{d-1}\}$$

is

$$J_d(x^2 - bx - c) = \begin{pmatrix} b & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & b & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & b & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & b & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & b & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c & 0 \end{pmatrix}$$

since $x \cdot p(x)^j x = p(x)^j x^2 = p(x)^j((bx+c+p(x)) = bp(x)^j x + cp(x)^j + p(x)^{j+1}$ and $x \cdot p(x)^j = p(x)^j x$.

Let $\mathbb{F}$ be a field and let $p(x) = x^r - a_{r-1}x^{r-1} - \cdots - a_1 x - a_0 \in \mathbb{F}[x]$. Let $r = \deg(p(x))$ and let $d \in \mathbb{Z}_{>0}$. The **d-Jordan block for $p(x)$** is

$$J_d(p(x)) = \begin{pmatrix} J_1(p(x)) & 0 & 0 & \cdots & 0 & 0 & 0 \\ E_{11} & J_1(p(x)) & 0 & \cdots & 0 & 0 & 0 \\ 0 & E_{11} & J_1(p(x)) & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & E_{11} & J_1(p(x)) & 0 \\ 0 & 0 & 0 & \cdots & 0 & E_{11} & J_1(p(x)) \end{pmatrix} \in M_{rd\times rd}(\mathbb{F})$$

where E_{11} is the $r \times r$ matrix with 1 in the $(1,1)$ entry and 0 elsewhere.

1.13.1 Connecting a matrix to an $\mathbb{F}[x]$-module

Proposition 1.49. *Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$ and let $A \in M_{n\times n}(\mathbb{F})$. The set $\mathbb{F}^n$ with the functions*

$$\begin{array}{ccc} \mathbb{F}^n \times \mathbb{F}^n & \to & \mathbb{F}^n \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathbb{F}[x] \times \mathbb{F}^n & \to & \mathbb{F}^n \\ (c_0 + c_1 x + \cdots + c_\ell x^\ell, v) & \mapsto & (c_0 + c_1 A \cdots + c_\ell A^\ell)v \end{array}$$

is an $\mathbb{F}[x]$-module.

Theorem 1.50. *(Jordan normal form) Let $\mathbb{F}$ be a field. Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_{n\times n}(\mathbb{F})$. Then there exist*

$$k \in \mathbb{Z}_{>0} \quad \textit{and} \quad d_1, \ldots, d_k \in \mathbb{Z}_{>0} \quad \textit{and} \quad p_1(x), \ldots, p_k(x) \in \mathbb{F}[x]$$

and an invertible matrix

$$P \in GL_n(\mathbb{F}) \qquad \textit{such that} \qquad PAP^{-1} = J_{d_1}(p_1(x)) \oplus \cdots \oplus J_{d_k}(p_k(x)).$$

Proof. Let $M = \mathbb{F}^n$ be the $\mathbb{F}[x]$ module given by Proposition 1.49. Write $A = (a_{ij})$ and let

$$K = A - x \in M_{n\times n}(\mathbb{F}[x]), \qquad \text{so that} \qquad K = (k_{ij}) \text{ with } \quad k_{ij} = \begin{cases} a_{ii} - x, & \text{if } i = j, \\ a_{ij}, & \text{if } i \neq j. \end{cases}$$

As and $\mathbb{F}[x]$ module, M is given by

$$\text{generators } \{e_1, \ldots, e_n\} \quad \text{and relations} \qquad \begin{array}{c} k_{11}e_1 + \cdots + k_{1n}e_n = 0, \\ \vdots \\ k_{n1}e_1 + \cdots + k_{nn}e_n = 0. \end{array}$$

Then there exist $P, Q \in GL_n(\mathbb{F}[x])$ and $d_1(x), \ldots, d_n(x) \in \mathbb{F}[x]$ such that M is presented by

$$\text{generators } \{b_1, \ldots, b_n\} \quad \text{and relations} \qquad d_1(x)b_1 = 0, \ \ldots, \ d_n(x)b_n = 0.$$

So

$$M \cong \frac{\mathbb{F}[x]}{d_1(x)\mathbb{F}[x]} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{d_n(x)\mathbb{F}[x]}$$

and the matrix of the action of x on M, with respect to a Jordan basis is in Jordan normal form. ☐