2.4 Proof of the relations between finiteness conditions

Proposition 2.5. *Let N be a submodule of M.*

(a) M satisfies ACC if and only if N and M/N satisfy ACC.

(b) M satisfies DCC if and only if N and M/N satisfy DCC.

(c) M satsfies both ACC and DCC if and only if N and M/N satisfy both ACC and DCC.

Proof. (a) $\Rightarrow$: Assume that M satisfies ACC.
To show: (aa) N satisfies ACC.
To show: (ab) M/N satsfies ACC.

(aa) Let $0 = N_0 \subseteq N_1 \subseteq \cdots$ be a chain in $\mathcal{S}_{[0,N]}$.
Since $N \subseteq M$ then $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq M$ is a chain in $\mathcal{S}_{[0,M]}$.
Since M satisfies ACC then $0 = N_0 \subseteq N_1 \subseteq \cdots$ is finite.
So N satisfies ACC.

(ab) Let $0 = M_0/N \subseteq M_1/N \subseteq \cdots \subseteq M/N$ be a chain in $\mathcal{S}_{[0,M/N]}$.
By the correspondence theorem the chain in $\mathcal{S}_{[0,M/N]}$ corresponds to a chain $0 \subseteq N = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ in $\mathcal{S}_{[0,M]}$.
Since M satsifes ACC then $0 \subseteq N = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ is finite.
So $0 = M_0/N \subseteq M_1/N \subseteq \cdots \subseteq M/N$ is finite.
So M/N satsfies ACC.

(a) $\Leftarrow$: Assume that N and M/N satisfy ACC.
To show: M satsifies ACC. Let $0 = M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain in $\mathcal{S}_{[0,M]}$.
Then

$$0 = \frac{M_0 + N}{N} \subseteq \frac{M_1 + N}{N} \subseteq \cdots \subseteq \frac{M}{N} \qquad \text{and} \qquad 0 = (M_0 \cap N) \subseteq (M_1 \cap N) \subseteq \cdots \subseteq N$$

are ascending chains in $\mathcal{S}_{[0,M/N]}$ and $\mathcal{S}_{[0,N]}$.
Let $k \in \mathbb{Z}_{>0}$ such that if $\ell \in \mathbb{Z}_{\geq k}$ then

$$\frac{M_\ell + N}{N} = \frac{M_k + N}{N} \qquad \text{and} \qquad M_\ell \cap N = M_k \cap N.$$

By the correspondence theorem, if $\ell \in \mathbb{Z}_{\geq k}$ then

$$M_\ell + N = M_k + N \qquad \text{and} \qquad M_\ell \cap N = M_k \cap N.$$

Thus

$$M_\ell \cap (M_k + N) = M_\ell \cap (M_\ell + N) = M_\ell \qquad \text{and} \qquad M_k + (m_\ell \cap N) = M_k + (M_k \cap N) = M_k.$$

Since $M_k \subseteq M_\ell$ then the modular law says that

$$M_\ell \cap (M_k + N) = M_k + (M_\ell \cap N).$$

So $M_k = M_\ell$.
(b) The proof of (b) is similar to the proof of (a),
except with ACC replaced by DCC and $\subseteq$ replaced by $\supseteq$.
(c) is the combination of (a) and (b). $\square$

Proposition 2.6. *Let R be a ring and let M be an R-module.*

(a) If M is finitely generated and N is an R-submodule of M then M/N is finitely generated.

(b) M satisfies ACC if and only if every submodule of M is finitely generated.

(c) If R satisfies ACC and M is finitely generated then M satisfies ACC.

(e) If R satisifes DCC and M is finitely generated then M satisfies both ACC and DCC.

Proof. (a) If $m_1, \ldots, m_k$ are generators of M then $m_1 + N, \ldots, m_k + N$ are generators of M/N.

(b) $\Leftarrow$: Assume that every submodule of M is finitely generated.
Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M.
To show: There exists $r \in \mathbb{Z}_{>0}$ such that if $\ell \in \mathbb{Z}_{\geq r}$ then $N_\ell = N_r$.
Then $N_{\text{un}} = \bigcup_{i\in\mathbb{Z}_{>0}} N_i$ is a finitely generated submodule of M
Let $x_1, \ldots, x_k$ be generators of N_{un} and let $\ell_1, \ldots, \ell_k$ be such that $x_i \in N_{\ell_i}$.
Then $x_1, \ldots, x_k \in N_r$ where $r = \max\{\ell_1, \ldots, \ell_k\}$.
So $N_{\text{un}} = \bigcup_{i\in\mathbb{Z}_{>0}} N_i = N_r$ and if $\ell \in \mathbb{Z}_{>r}$ then $N_r = N_\ell$.
So M satisfies ACC.

(b) $\Rightarrow$: Assume that M satisfies ACC and let N be a submodule of M. Then one of the equivalent characterizations of ACC gives that the set of finitely generated submodule of N,

$$\{P \subseteq N \mid P \text{ is finitely generated}\}, \qquad \text{has a maximal element } P_{\text{max}}.$$

To show: $N = P_{\text{max}}$.
By definition, $P_{\text{max}} \subseteq N$.
To show: $N \subseteq P_{\text{max}}$.

Let $x \in N$.

Then $P_{\text{max}} + \mathbb{A}x \subseteq N$ and $P_{\text{max}} + \mathbb{A}x$ is finitely generated.

So $P_{\text{max}} + \mathbb{A}x \subseteq P_{\text{max}}$.

So $x \in P_{\text{max}}$.

So $N \subseteq P_{\text{max}}$.

So $N = P_{\text{max}}$.
So N is finitely generated.

(c) $\Rightarrow$: Assume R satisfies ACC and M is finitely generated.
To show: M satsfies ACC.
Since M is finitely generated there exists $n \in \mathbb{Z}_{>0}$ and a surjective homomorphism $\mathbb{A}^{\oplus v} \to M$.
Since $\mathbb{A}$ satisfies ACC then $\mathbb{A}^{\oplus n}$ satisifes ACC.
So there is an exact sequence $0 \to K \to \mathbb{A}^{\oplus n} \to M \to 0$ with $\mathbb{A}^{\oplus n}$ satisfiying ACC.
By Proposition 4.4(a), K and M satisfy ACC.
So M satisfies ACC.

(da) To show: If R satisfies DCC and M is finitely generated then M satisfies DCC. The proof of (da) is the same as the proof of (c) except with ACC replaced by DCC and the inclreasing chains replaced by decreasing chains.

(db) Assume R satisfies DCC and M is finitely generated. To show: M satisfies ACC.
Let $M_i = \text{Rad}(R)^i M$.
By (da), M satisfies DCC, and so M_i and M/M_i satisfy DCC. So M_i/M_{i+1} satisfies DCC and $\text{Rad}(R)$ acts on M_i/M_{i+1} by 0.
So M_i/M_{i+1} is a $R/\text{Rad}(R)$-module and thus M_i/M_{i+1} is a finite direct sum of simple submodules.
So, by (a), M has a composition series and satisfies both ACC and DCC. $\square$