18.10.2023 (1)

Linear algebra

A. Raw

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x,y,z) = (x+y,\ x+2y-z,\ 2x+y+z).$$

Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ and

$B = \{(1,0,1), (-2,1,1), (1,-1,1)\}$.

Then

$$b_1 = (1,0,1) = 1\cdot(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)$$

$$b_2 = (-2,1,1) = -2\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1)$$

$$b_3 = (1,-1,1) = 1\cdot(1,0,0) + (-1)(0,1,0) + 1\cdot(0,0,1)$$

So

$$P_{SB} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{Then}$$

$$P_{BS} = P_{SB}^{-1}$$

(b) Since

$$T(1,0,0) = (1,1,2) = 1\cdot(1,0,0) + 1\cdot(0,1,0) + 2\cdot(0,0,1)$$

$$T(0,1,0) = (1,2,1) = 1\cdot(1,0,0) + 2(0,1,0) + 1\cdot(0,0,1)$$

$$T(0,0,1) = (0,-1,1) = 0\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1)$$

Then

$$[T]_{SS} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Then

$$[T]_{BB} = P_{BS}[T]_{SS}P_{SB}$$

(2) $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$$A - t = \begin{pmatrix} 4-t & 0 & -1 \\ 0 & 3-t & 0 \\ 1 & 0 & 2-t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4-t & 0 & -1 \\ 1 & 0 & 2-t \\ 0 & 3-t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4-t & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2-t \\ 0 & 0 & -1-(4-t)(2-t) \\ 0 & 3-t & 0 \end{pmatrix}$$

$$= y_2(0)\, y_1(4-t)\begin{pmatrix} 1 & 0 & 2-t \\ 0 & 0 & -t^2+6t-9 \\ 0 & 3-t & 0 \end{pmatrix}$$

$$= y_2(0)\, y_1(4-t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2-t \\ 0 & 3-t & 0 \\ 0 & 0 & -(t-3)^2 \end{pmatrix}$$

$$= y_2(0)\, y_1(4-t)\, y_2(0)\begin{pmatrix} 1 & 0 & 2-t \\ 0 & 3-t & 0 \\ 0 & 0 & -(t-3)^2 \end{pmatrix}$$

When $t=3$

$$\ker(A-3) = \ker\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$