16.10.2023
Linear Algebra
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An <u>orthogonal matrix</u> is $Q \in M_n(\mathbb{R})$ such that
$$Q^t Q = 1$$
A <u>unitary matrix</u> is $U \in M_n(\mathbb{R})$ such that
$$\bar{U}^t U = 1$$
A <u>symmetric matrix</u> is $A \in M_n(\mathbb{R})$ such that
$$A^t = A$$
A <u>Hermitian matrix</u> is $A \in M_n(\mathbb{C})$ such that
$$\bar{A}^t = A.$$

<u>Example 21</u> $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ is Hermitian since
$$\bar{A}^t = \begin{pmatrix} \bar{1} & \bar{-i} \\ \bar{i} & \bar{1} \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A$$
$B = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is not Hermitian since
$$\bar{B}^t = \begin{pmatrix} \bar{i} & \bar{0} \\ \bar{0} & \bar{i} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \neq B.$$

<u>Example 20</u> $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary since
$$\bar{U}^t U = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{-i} & \bar{1} \\ \bar{i} & \bar{1} \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

16.10.2023
Linear Algebra 2
A. Ram

<u>Example 15</u> $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal

since

$$Q^tQ = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example 16</u> Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$.

Then

$$\langle \vec{u} | \vec{v} \rangle = \vec{u}^t \vec{v} \quad \text{and}$$

$$\langle Q\vec{u} | Q\vec{v} \rangle = (Q\vec{u})^t Q\vec{v} = \vec{u}^t Q^t Q \vec{v} = \vec{u}^t 1 \cdot \vec{v} = \vec{u}^t \vec{v}$$

So $\langle Q\vec{u} | Q\vec{v} \rangle = \langle \vec{u} | \vec{v} \rangle$.

<u>Example 21</u> $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ is Hermitian.

$$\ker(A-t) = \ker\begin{pmatrix} 1-t & i \\ i & 1-t \end{pmatrix} = \ker\begin{pmatrix} -i & 1-t \\ 0 & i(t^2-2t+1-1) \end{pmatrix}$$

since

$$\begin{pmatrix} 1-t & i \\ -i & 1-t \end{pmatrix} = \begin{pmatrix} (1-t)i & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i & 1-t \\ 0 & i(1-t)^2 \end{pmatrix}$$

If $t=0$ then $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector.

and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector of length 1.

If $t=2$ then $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector

and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector of length 1.

Then

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ is unitary and}$$

$$A=U\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\bar{U}^t.$$

<u>Example 18</u> $A=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is symmetric

$$\ker\begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix}=\ker\begin{pmatrix} -1 & 1-t \\ 0 & t^2-2t \end{pmatrix}$$

since

$$\begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix}=\begin{pmatrix} -(1-t) & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1-t \\ 0 & -1+(1-t)^2 \end{pmatrix}$$

If $t=0$ then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of length 1

If $t=2$ then $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of length 1.

then $Q = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ is orthogonal

and
$$A = Q\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}Q^t$$

Check:
$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}0 & 2\\ 0 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$= \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = A.$$

Examples 3, 4 and 10 Topic 6

$$A = \begin{pmatrix}2 & -3 & 6\\ 0 & 5 & 6\\ 0 & 1 & 0\end{pmatrix}$$

then

$$\ker\begin{pmatrix}2-t & -3 & 6\\ 0 & 5-t & 6\\ 0 & 1 & -t\end{pmatrix} = \ker\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 5-t & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}2-t & -3 & 6\\ 0 & 1 & -t\\ 0 & 0 & -6-(5-t)(-t)\end{pmatrix}\right)$$

$$= \ker\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 5-t & 1\\ 0 & 1 & 0\end{pmatrix}\overset{\begin{pmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}{\begin{pmatrix}2-t & 0 & 3(2-t)\\ 0 & 1 & -t\\ 0 & 0 & -(t^2-5t+6)\end{pmatrix}}\right)$$

$$= \ker\begin{pmatrix}2-t & 0 & 3(2-t)\\ 0 & 1 & -t\\ 0 & 0 & -(t-3)(t-2)\end{pmatrix}$$

and this has a row of 0's if $t = 3$

and two rows of 0s when $t = 2$.

16.10.2023
Linear Algebra
5
A. Ram

If $t=2$ then $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector

and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector.

If $t=3$ then

$\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector.

Let $P=\begin{pmatrix} 0 & 1 & -3 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix}$ and $D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Then

$A=PDP^{-1}$.