Orthogonal matrices and
Singular value decomposition

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Linear Algebra ①
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Consider $\mathbb{R}^n$ with the standard inner product

$$\langle \vec{u}, \vec{v}\rangle = \vec{u}^t\vec{v} = (u_1\ u_2 \cdots u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= u_1v_1 + \cdots + u_nv_n, \text{ when } \vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Let $A \in M_{n\times n}(\mathbb{R})$. The matrix A is orthogonal if

$$AA^t = 1 \quad \text{and} \quad A^tA = 1.$$

The columns of A form a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}.$$

then

$$A^tA = \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$$

means $\langle \vec{v}_i, \vec{v}_j\rangle = \delta_{ij}$.

So the columns of A form an orthonormal basis if and only if

A is an orthogonal matrix.

Let $O_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^tA = 1\}$

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Singular value decomposition

$$A \in M_{t\times s}(\mathbb{R})$$

Find

$$U \in O_t(\mathbb{R}), \quad V \in O_s(\mathbb{R}), \quad S \in M_{t\times s}(\mathbb{R})$$

with S "diagonal" and

$$A = USV^t$$

Since $V \in O_s(\mathbb{R})$ then

$A = USV^t$ is equivalent to $AV = US$.

S is "diagonal" means

$S = \sigma_1 E_{11} + \cdots + \sigma_r E_{rr}$ where $r = \min(s,t)$,

and $E_{ij} \in M_{t\times s}(\mathbb{R})$ is the matrix with 1 in the (i,j) entry and 0 elsewhere,

and $\sigma_1, \ldots, \sigma_r \in \mathbb{R}$. So

$$\begin{pmatrix} & A & \end{pmatrix} \underbrace{\begin{pmatrix} & & \\ & V & \\ & & \end{pmatrix}}_{\text{orthogonal}} = \underbrace{\begin{pmatrix} & U & \end{pmatrix}}_{\text{orthogonal}} \begin{pmatrix} \sigma_1 & & & \\ 0 & \sigma_2 & & 0 \\ & & \sigma_3 & \end{pmatrix}$$

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then

$$\begin{pmatrix} \\ A^t \\ \\ \end{pmatrix} \begin{pmatrix} & A & \end{pmatrix} = \begin{pmatrix} & & \\ & A^tA & \\ & & \end{pmatrix}$$

and the columns of V are orthonormal eigenvectors of A^tA.

Then

$$\begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \\ A^t \\ \\ \end{pmatrix} = \begin{pmatrix} AA^t \end{pmatrix}$$

and the columns of U are orthonormal eigenvectors of AA^t.

If $\lambda_1, \dots, \lambda_s$ are the eigenvalues of A^tA then $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots, \sigma_r = \sqrt{\lambda_r}$

If $\vec{v}_1, \dots, \vec{v}_s$ are the columns of V then

$$\vec{u}_1 = \frac{1}{\sigma_1}\vec{v}_1, \dots, \vec{u}_r = \frac{1}{\sigma_r}\vec{v}_r$$

are the first r columns of U. (the remaining columns $\vec{u}_{r+1}, \dots, \vec{u}_k$ can be any vectors so that

$\{\vec{u}_1, \dots, \vec{u}_k\}$ is an orthonormal basis

(use Gram-Schmidt to get orthonormal sets).

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Example 19 $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

Find $U \in O_2(\mathbb{R})$ and $V \in O_2(\mathbb{R})$ and

$$S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \text{ so that } A = USV^t.$$

The columns of V should be eigenvectors of A^tA.

$$A^tA = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which has eigenvectors

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue $\lambda_1 = 0$

$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue $\lambda_2 = 1$.

So $\sigma_1 = \sqrt{0}$ and $\sigma_2 = \sqrt{1}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

and $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then

$$AV = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ since $U^tU = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$