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Linear algebra
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Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$.

Let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle : V \times V \to \mathbb{F}$

An <u>orthonormal set in V</u> is a subset $S=\{v_1, v_2, \ldots, v_k\}$ of V such that

if $i,j \in \{1, \ldots, k\}$ and $i \neq j$ then $\langle v_i, v_j \rangle = 0$.

<u>Proposition</u> If S is an orthonormal set then S is linearly independent.

<u>Proof</u> Assume $S=\{v_1, v_2, \ldots, v_k\}$ is an orthonormal set.

To show: If $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1v_1 + \cdots + c_kv_k = 0$ then $c_1=0, c_2=0, \ldots, c_k=0$.

Assume $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1v_1 + \cdots + c_kv_k = 0$.

To show: If $j \in \{1, \ldots, k\}$ then $c_j = 0$.

Assume $j \in \{1, \ldots, k\}$.

To show: $0 = c_j$.

$$
\begin{aligned}
0 &= \langle c_1v_1 + \cdots + c_kv_k, v_j \rangle \\
&= c_1\langle v_1, v_j \rangle + \cdots + c_{j-1}\langle v_{j-1}, v_j \rangle + c_j \langle v_j, v_j \rangle \\
&\quad + c_{j+1}\langle v_{j+1}, v_j \rangle + \cdots + c_k \langle v_k, v_j \rangle
\end{aligned}
$$

$= c_1 \cdot 0 + \cdots + c_{j-1} \cdot 0 + c_j \langle v_j, v_j \rangle$
$+ c_{j+1} \cdot 0 + \cdots + c_k \cdot 0$

$= c_j \langle v_j, v_j \rangle.$

Since $0 = c_j \langle v_j, v_j \rangle$ and $\langle v_j, v_j \rangle \neq 0$ then $c_j = 0$

So S is linearly independent.

An orthonormal basis of V is a subset $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ such that

(a) B is an orthonormal subset of V
(b) B is a basis of V.

Proposition Assume $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is an orthonormal basis of V and $\vec{x} \in V$. Then

$$\vec{x} = \langle \vec{x}, \vec{b}_1 \rangle \vec{b}_1 + \cdots + \langle \vec{x}, \vec{b}_n \rangle \vec{b}_n .$$

Orthogonal projections

Let W be a subspace of V.

Let $\{\vec{b}_1, \ldots, \vec{b}_k\}$ be an orthonormal basis of W.

Let $\vec{x} \in V$. The <u>orthogonal projection of $\vec{x}$ onto W</u> is

$$\operatorname{proj}_W(\vec{x}) = \langle \vec{x}, \vec{b}_1 \rangle \vec{b}_1 + \cdots + \langle \vec{x}, \vec{b}_k \rangle \vec{b}_k$$

<u>Example</u> Let $V = \mathbb{R}^n$ and let $\vec{u} \in V$ with $\vec{u} \neq \vec{0}$. Let

$$W = \operatorname{span}\{\vec{u}\} = \{ a\vec{u} \mid a \in \mathbb{R} \}.$$

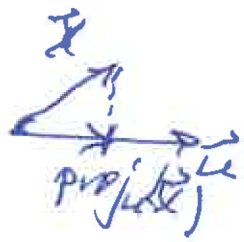
Then W is a 1 dimensional subspace of V.

Let $\vec{b}_1 = \frac{1}{\|\vec{u}\|}\vec{u}$. Then $\{\vec{b}_1\}$ is an orthonormal basis of W.

Let $\vec{x} \in V$. Then

$$\operatorname{proj}_W(\vec{x}) = \langle \vec{x}, \vec{b}_1 \rangle \vec{b}_1 = \left\langle \vec{x}, \frac{\vec{u}}{\|\vec{u}\|} \right\rangle \frac{\vec{u}}{\|\vec{u}\|}$$

$$= \frac{\langle \vec{x}, \vec{u} \rangle}{\|u\|^2} \vec{u} = \operatorname{proj}_{\vec{u}}(\vec{x})$$



Example 12 Let $V = \mathbb{R}^3$ and let

$$W = \{ \langle x, y, z \rangle \in \mathbb{R}^3 \mid x+y+z = 0 \}.$$

The set

$$\{b_1, b_2\} = \{ \tfrac{1}{\sqrt{2}}(1,-1,0), \tfrac{1}{\sqrt{6}}(1,1,-2) \}$$

is an orthonormal basis of W with respect to the standard inner product.

Let $\vec{x} = \langle 1, 2, 3 \rangle$.

Then

$$\mathrm{proj}_W(\vec{x}) = \langle \vec{x}, \vec{b}_1 \rangle \vec{b}_1 + \langle \vec{x}, \vec{b}_2 \rangle \vec{b}_2$$

$$= \langle 1,2,3 \mid \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0 \rangle \tfrac{1}{\sqrt{2}} \langle 1,-1,0 \rangle$$

$$+ \langle 1,2,3 \mid \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}} \rangle \tfrac{1}{\sqrt{6}} \langle 1,1,-2 \rangle$$

$$= \left(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} \right) \tfrac{1}{\sqrt{2}} (1,-1,0) + \tfrac{1}{6}(1+2-6) \langle 1,1,-2 \rangle$$

$$= \langle -\tfrac{1}{2}, \tfrac{1}{2}, 0 \rangle + \langle -\tfrac{1}{2}, -\tfrac{1}{2}, 1 \rangle = \langle -1, 0, 1 \rangle$$

The shortest distance from $\vec{x}$ to W is

$$\| \vec{x} - \mathrm{proj}_W(\vec{x}) \| = \| \langle 1,2,3 \rangle - \langle -1,0,1 \rangle \|$$

$$= \| \langle 2, +2, 2 \rangle \| = \sqrt{4+4+4} = 2\sqrt{3}.$$