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Linear Algebra given by A. Rau

Example 2 $\langle\ ,\ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3$$

$$= (u_1,u_2,u_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

has

$$\langle (0,1,0),(0,1,0)\rangle = 0-1+0 = -1 \notin \mathbb{R}_{\geq 0}.$$

So this is not positive definite.

Example 6 $\langle\ ,\ \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = i u_1\bar{v}_1 - i u_2 \bar{v}_2$$

$$= (u_1,u_2)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

has

$$\langle (1,0),(1,0)\rangle = i\cdot 1\cdot \bar{1} - i\cdot 0\cdot \bar{0} = i \notin \mathbb{R}_{\geq 0}.$$

So this is not positive definite.

Example 3 $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + 3u_2v_2$$

$$= (u_1\ u_2)\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

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has $\langle (u_1,u_2),(u_1,u_2)\rangle = 2u_1^2 - 2u_1u_2 - 2u_2u_1 + 3u_2^2$

$$= 2u_1^2 - 4u_1u_2 + 3u_2^2$$

$$= 2(u_1^2 - 2u_1u_2 + u_2^2) + u_2^2$$

$$= 2(u_1-u_2)^2 + u_2^2 \in \mathbb{R}_{\ge 0}$$

and if $2(u_1-u_2)^2 + u_2^2 = 0$ then $u_2^2=0$ and $(u_1-u_2)^2=0$ so that $u_2=0$ and $u_1-u_2=0$.

So if $\langle (u_1,u_2),(u_1,u_2)\rangle = 0$ then $(u_1,u_2)=(0,0)$.

So this is positive definite.

<u>Example</u> $\langle\, ,\rangle : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2$$

$$= (u_1\ u_2)\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

has $\langle (u_1,u_2),(u_1,u_2)\rangle = u_1^2 + 2u_2^2 \in \mathbb{R}_{\ge 0}$

and if $u_1^2 + 2u_2^2 = 0$ then $u_1^2=0$ and $u_2^2=0$.

So, if $\langle (u_1,u_2),(u_1,u_2)\rangle = 0$ then $(u_1,u_2)=(0,0)$.

So this is positive definite.

Example 8 If V is an $\mathbb{F}$-vector space with an inner product and $u, v \in V$ and u and v are orthogonal then

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$
$$= \langle u, u+v\rangle + \langle v, u+v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + 0 + 0 + \|v\|^2$$
$$= \|u\|^2 + \|v\|^2.$$

this is the Pythagorean theorem.

Example Let $\langle\, , \,\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by

$$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n)\rangle = (u_1 \ldots u_n) A \begin{pmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{pmatrix} = u^t A \bar{v}$$

with A satisfying $A = \bar{A}^t$.

Then

$$\langle v, u\rangle = \overline{(\overline{v^t A \bar{u}})} = \overline{(\bar{u}^t A^t v)^t} = (u^t \bar{A}^t \bar{v})^t$$
$$= (u^t A \bar{v})^t = \langle u, v\rangle$$

and

$$\langle \alpha u, v\rangle = (\alpha u)^t A \bar{v} = \alpha u^t A v = \alpha \langle u, v\rangle$$

and

$$\langle u+v, w\rangle = (u+v)^t A\bar{w}$$
$$= (u^t+v^t)A\bar{w}$$
$$= u^tA\bar{w} + v^tA\bar{w}$$
$$= \langle u,w\rangle + \langle v,w\rangle.$$

So $\langle , \rangle$ satisfies all the properties of an inner product, except perhaps not positive definiteness.

<u>Gram matrix</u>

Let V be an $\mathbb{F}$-vector space with inner product $\langle , \rangle : V\times V \to \mathbb{F}$.

Let $B=\{b_1,\ldots,b_n\}$ be a basis of V.
The <u>Gram matrix</u> of $\langle , \rangle$ with respect to B is the matrix

$A \in M_{n\times n}(\mathbb{F})$ with (i,j) entry $a_{ij} = \langle b_i, b_j\rangle$

Let $u = u_1b_1 + \cdots u_nb_n \in V$ so that

$$[u]_B = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Let $v = v_1 b_1 + \cdots + v_n b_n \in V$ so that

$$[v]_B = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Then

$$\begin{aligned}\langle u, v\rangle &= \langle u_1 b_1 + \cdots + u_n b_n, v_1 b_1 + \cdots + v_n b_n\rangle \\ &= \sum_{i,j=1}^{n} u_i v_j \langle b_i, b_j\rangle \\ &= \sum_{i,j=1}^{n} u_i a_{ij} v_j = [u]_B^t A [v]_B.\end{aligned}$$