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Linear Algebra
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Example 21 $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$$\det(A-t) = \begin{vmatrix} 1-t & i \\ -i & 1-t \end{vmatrix} = (1-t)^2 - (-i)i = 1-2t+t^2-1$$

$$= t^2-2t = t(t-2)$$

So $\det(A-t)=0$ when $t=0$ or $t=2$.

$$\ker(A-0) = \ker\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

$$= \operatorname{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\} \text{ since } \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\ker(A-2) = \ker\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix} = \ker\begin{pmatrix} -1 & i \\ 0 & 0 \end{pmatrix}$$

$$= \operatorname{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\} \text{ since } \begin{pmatrix} -1 & i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $\|\langle -i, 1\rangle\|^2 = (-i)\overline{(-i)} + 1\cdot\overline{1} = (-i)i + 1\cdot 1 = 1+1 = 2.$

So $\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is a unit vector

Then $\|\langle i, 1\rangle\|^2 = i\bar{i} + 1\cdot\bar{1} = i(-i)+1 = 1+1 = 2$

So $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ is a unit vector.

Let $P = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

Example 17 $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Then

$$AA^t = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^tA = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A^tA has eigenvectors

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of eigenvalue 0 and

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of eigenvalue 1.

Example 17 $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. (symmetric).

$$\det(A-t) = \det\begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix} = (1-t)^2 - 1 = -2t + t^2$$

$$= t(t-2)$$

So $\det(A-t) = 0$ when $t=0$ or $t=2$

$$\ker(A-0) = \ker\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

and

$$\ker(A-2) = \ker\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \ker\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \mathrm{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}.$$

Then $\|(1,1)\|^2 = 1^2+1^2 = 2$ and

$\|(1,-1)\|^2 = 1^2+(-1)^2 = 2$

So

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ are unit vectors that are eigenvectors of A.