Example 15 Pink, red and white snapdragon plants.

(1) $P \heartsuit R = \frac{1}{2}R + \frac{1}{2}P$

(2) $P \heartsuit P = \frac{1}{4}R + \frac{1}{2}P + \frac{1}{4}W$

(3) $P \heartsuit W = \frac{1}{2}P + \frac{1}{2}W$.

Cross continually with pink plants.

Let

$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix}, \quad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and } x_{n+1} = Tx_n$$

Then

$$T = PDP^{-1}$$

where

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Then

$$\lim_{n\to\infty} T^n = \lim_{n\to\infty} PD^nP^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}P^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

So

$$\lim_{n\to\infty} T^n x_0 = \begin{pmatrix} \frac{1}{4}(r_0 + p_0 + w_0) \\ \frac{1}{2}(r_0 + p_0 + w_0) \\ \frac{1}{4}(r_0 + p_0 + w_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\cdot 1 \\ \frac{1}{2}\cdot 1 \\ \frac{1}{4}\cdot 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

is the stationary distribution (independent of starting distribution).

Example Eigenvalues of $D = \frac{d}{dx}$

Let $\mathbb{R}[[x]] = \{a_0 + a_1 x + \cdots \mid a_i \in \mathbb{R}\}$

so that
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \in \mathbb{R}[[x]]$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \in \mathbb{R}[[x]]$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \in \mathbb{R}[[x]]$$

Let $D: \mathbb{R}[[x]] \to \mathbb{R}[[x]]$ be the linear transformation given by
$$D(p) = \frac{dp}{dx}.$$

Find the eigenvalues and eigenvectors of D.

Let $p = a_0 + a_1 x + a_2 x^2 + \cdots$

Let $\lambda \in \mathbb{R}$. Then
$$(D-\lambda)(p) = 0 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$- \lambda a_0 - \lambda a_1 x - \lambda a_2 x^2 + \cdots$$

gives
$$a_1 - \lambda a_0 = 0, \qquad\qquad a_1 = \lambda a_0$$
$$2a_2 - \lambda a_1 = 0, \quad \text{so} \quad a_2 = \frac{1}{2}\lambda a_1 = \frac{1}{2}\lambda^2 a_0$$
$$3a_3 - \lambda a_2 = 0, \qquad\qquad a_3 = \frac{1}{3}\lambda a_2 = \frac{1}{3!}\lambda^3 a_0$$
$$\vdots \qquad\qquad\qquad \vdots$$

So $p = a_0 e^{\lambda x}$.

So $\ker(D-\lambda) = \operatorname{span}\{e^{\lambda x}\}$.

Relations for $\ker(A)$ and $\operatorname{im}(A)$

18.09.2023 (3)
Linear Algebra
A. Lain

$$\ker(P^{-1}AQ^{-1}) = \{v \in \mathbb{F}^n \mid P^{-1}AQ^{-1}v = 0\}$$

$$= \{v \in \mathbb{F}^n \mid AQ^{-1}v = P \cdot 0\}$$

$$= \{QQ^{-1}v \in \mathbb{F}^n \mid AQ^{-1}v = 0\}$$

$$= \{Qw \in \mathbb{F}^n \mid Aw = 0\}$$

$$= Q \cdot \{w \in \mathbb{F}^n \mid Aw = 0\}$$

$$= Q \ker(A)$$

and

$$\operatorname{im}(P^{-1}AQ^{-1}) = \{P^{-1}AQ^{-1}v \mid v \in \mathbb{F}^n\}$$

$$= P^{-1}\{AQ^{-1}v \mid v \in \mathbb{F}^n\}$$

$$= P^{-1}\{AQ^{-1}v \mid QQ^{-1}v \in \mathbb{F}^n\}$$

$$= P^{-1}\{Aw \mid Qw \in \mathbb{F}^n\}$$

$$= P^{-1}\{Aw \mid w \in \mathbb{F}^n\} = P^{-1}\operatorname{im}(A).$$

Solving equations

18.09.2023

Linear algebra

A. Rain

(4)

$A = PR$

$$= P \begin{pmatrix} 1 & a_{1j_1} & a_{1j_2} & 0 & a_{1j_3} & 0 \\ 0 & 0 & 0 & 1 & a_{2j_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then

$$\ker(A) = \ker \begin{pmatrix} 1 & a_{1j_1} & a_{1j_2} & 0 & a_{1j_3} & 0 \\ 0 & 0 & 0 & 1 & a_{2j_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$c_{i_1} + a_{1j_1} c_{j_1} + a_{1j_2} c_{j_2} + a_{1j_3} c_{j_3} = 0$$

$$c_{i_2} + a_{2j_3} c_{j_3} = 0$$

$$c_{i_3} = 0.$$

So

$$\ker(A) = \operatorname{span}\left\{ c_{j_1} \begin{pmatrix} -a_{1j_1} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_{j_2} \begin{pmatrix} -a_{1j_2} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_{j_3} \begin{pmatrix} -a_{1j_3} \\ a_{2j_3} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Example 18

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

and $A = QDQ^t$.

Columns of Q are eigenvectors of A.

Example 22

$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

Columns of U are eigenvectors of A

Orthogonal $AA^t = I$

Unitary $A\bar{A}^t = I$

Symmetric $A = A^t$.

Hermitian $A = \bar{A}^t$.

<u>Example</u>

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

then

$$\det(A-t) = \det \begin{pmatrix} 2-t & 1 & & & \\ & 2-t & 1 & & \\ & & 2-t & & \\ & & & 6-t & 1 \\ & & & & 6-t \end{pmatrix}$$

$$= (2-t)^3(6-t)^2$$

So $\det(A-t) = 0$ when $t=2$ or $t=6$

$$\ker(A-2) = \ker \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\ker(A-6) = \ker \begin{pmatrix} -4 & 1 & & & \\ & -4 & 1 & & \\ & & -4 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

So A is <u>not</u> diagonalizable

Eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Example
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 6 & 2 & 8 & 1 & 4 \end{pmatrix}$$

Then

$$\det(A-t) = \begin{pmatrix} -t & 1 & 0 & 0 & 0 \\ 0 & -t & 1 & 0 & 0 \\ 0 & 0 & -t & 1 & 0 \\ 0 & 0 & 0 & -t & 1 \\ 6 & 2 & 8 & 1 & 4-t \end{pmatrix}$$

$$= 6\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix} - 2\det\begin{pmatrix} -t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}$$

$$+ 8\det\begin{pmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix} - \det\begin{pmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ (4-t)\det\begin{pmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & 1 \\ 0 & 0 & 0 & -t \end{pmatrix}$$

$$= 6 + 2t + 8t^2 + t^3 + 4t^4 - t^5.$$