Examples 2 and 6 and 9

Let
$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A - t = \begin{pmatrix} 1-t & 4 \\ 1 & 1-t \end{pmatrix}$$

Then
$$\det(A-t) = (1-t)^2 - 4 = 1 - 2t + t^2 - 4$$
$$= t^2 - 2t - 3 = (t-3)(t+1).$$

So
$$\det(A-3) = 0 \quad \text{and} \quad \det(A+1) = 0$$

So
$$\ker(A-3) \neq 0 \quad \text{and} \quad \ker(A+1) \neq 0.$$

Then
$$\ker(A-3) = \ker\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \ker\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$$
$$= \mathrm{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \quad \text{since} \quad \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and
$$\ker(A+1) = \ker\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \ker\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$
$$= \mathrm{span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\} \quad \text{since} \quad \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So
$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{since} \quad (A-3)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{since} \quad (A+1)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $AP = PD$ where $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

So

$$P^{-1}AP = D$$

Then

$$tr(A) = tr(D) = 3-1 = 2 \text{ and}$$

$$\det(A) = \det(D) = 3\cdot(-1) = -3.$$

<u>Example 5 and 8 and 12</u>

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } A - t = \begin{pmatrix} -t & 1 \\ -1 & -t \end{pmatrix}.$$

Then

$$\det(A-t) = (-t)^2 - (-1) = t^2+1.$$

If $t \in \mathbb{R}$ then $\det(A-t)$ is never 0.

So A has no $\mathbb{R}$-eigenvalues and no $\mathbb{R}$-eigenvectors.

If $t \in \mathbb{C}$ and $\det(A-t) = 0$ then

$0 = t^2+1 = (t+i)(t-i)$ and $t = i$ or $t = -i$.

So A has 2 $\mathbb{C}$-eigenvalues, i and $-i$.

So $\det(A-i)=0$ and $\det(A+i)=0$

So $\ker(A-i)\neq 0$ and $\ker(A+i)\neq 0$.

Then

$$\ker(A-i)=\ker\begin{pmatrix}-i & 1\\ -1 & -i\end{pmatrix}=\ker\begin{pmatrix}0 & 0\\ -1 & -i\end{pmatrix}=\ker\begin{pmatrix}0 & 0\\ 1 & i\end{pmatrix}$$

$$=\mathbb{C}\text{-span}\left\{\begin{pmatrix}-i\\ 1\end{pmatrix}\right\} \text{ since } \begin{pmatrix}0 & 0\\ 1 & i\end{pmatrix}\begin{pmatrix}-i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

and

$$\ker(A+i)=\ker\begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}=\ker\begin{pmatrix}0 & 0\\ -1 & i\end{pmatrix}=\ker\begin{pmatrix}0 & 0\\ 1 & -i\end{pmatrix}$$

$$=\mathbb{C}\text{-span}\left\{\begin{pmatrix}i\\ 1\end{pmatrix}\right\} \text{ since } \begin{pmatrix}0 & 0\\ 1 & -i\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

So

$$A\begin{pmatrix}-i\\ 1\end{pmatrix}=i\begin{pmatrix}-i\\ 1\end{pmatrix} \text{ since } (A-i)\begin{pmatrix}-i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$A\begin{pmatrix}i\\ 1\end{pmatrix}=-i\begin{pmatrix}i\\ 1\end{pmatrix} \text{ since } (A+i)\begin{pmatrix}i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

So $\begin{pmatrix}-i\\ 1\end{pmatrix}$ is a $\mathbb{C}$-eigenvector of A with $\mathbb{C}$-eigenvalue i

and $\begin{pmatrix}i\\ 1\end{pmatrix}$ is a $\mathbb{C}$-eigenvector of A with $\mathbb{C}$-eigenvalue $-i$.

Examples 13 and 14

If
$$D = \begin{pmatrix}-4 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} \quad \text{then } D^{10} = \begin{pmatrix}(-4)^{10} & 0 & 0\\ 0 & 3^{10} & 0\\ 0 & 0 & 2^{10}\end{pmatrix}$$

If $A = PDP^{-1}$ then

$$A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$= PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^3P^{-1}$$

and $\quad A^k = PD^kP^{-1}$ for $k \in \mathbb{Z}$

Take
$$A = \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix}, \quad P = \begin{pmatrix}-2 & 2\\ 1 & 1\end{pmatrix}, \quad D = \begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}$$

so that
$$P^{-1} = \frac{1}{4}\begin{pmatrix}-1 & 2\\ 1 & 2\end{pmatrix} \quad \text{and } P^{-1}AP = D$$

Then
$$A^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1} = \begin{pmatrix}-2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}(-1)^{10} & 0\\ 0 & 3^{10}\end{pmatrix}\frac{1}{4}\begin{pmatrix}-1 & 2\\ 1 & 2\end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix}(-1)^{10}(-2) & 3^{10}\cdot 2\\ (-1)^{10} & 3^{10}\end{pmatrix}\begin{pmatrix}-1 & 2\\ 1 & 2\end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix}2((-1)^{10}+3^{10}) & 4((-1)^{10+1}+3^{10})\\ (-1)^{10+1}+3^{10} & 2((-1)^{10}+3^{10})\end{pmatrix}.$$

Let $A \in M_{n \times n}(\mathbb{F})$.

The matrix A is <u>$\mathbb{F}$-diagonalisable</u> if there exists $P \in GL_n(\mathbb{F})$ such that

$$P^{-1}AP = D, \text{ with } D \text{ diagonal}.$$

The <u>characteristic polynomial</u> of A is

$$\det(A - t).$$

An <u>$\mathbb{F}$-eigenvalue of A</u> is $\lambda \in \mathbb{F}$ such that

$$\ker(A - \lambda) \neq 0.$$

An <u>$\mathbb{F}$-eigenvector of A of eigenvalue λ</u> is a nonzero $v \in \mathbb{F}^n$ such that $v \in \ker(A - \lambda)$.

<u>Note:</u> $P \in GL_n(\mathbb{F})$ satisfies

$$P^{-1}AP = D, \text{ with } D \text{ diagonal}$$

if and only if

the columns of P are linearly independent $\mathbb{F}$-eigenvectors of A