Eigenvalues and Eigenvectors

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Linear Algebra ①
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Example 3 and 4 and 10

$$A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}$$

So $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is an eigenvector of A of eigenvalue 2.

$$A\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\4\\2\end{pmatrix} = 2\begin{pmatrix}0\\2\\1\end{pmatrix}$$

So $\begin{pmatrix}0\\2\\1\end{pmatrix}$ is an eigenvector of A of eigenvalue 2.

$$A\begin{pmatrix}-3\\3\\1\end{pmatrix} = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}-3\\3\\1\end{pmatrix} = \begin{pmatrix}-9\\9\\3\end{pmatrix} = 3\begin{pmatrix}-3\\3\\1\end{pmatrix}$$

So $\begin{pmatrix}-3\\3\\1\end{pmatrix}$ is an eigenvector of A of eigenvalue 3

So

$$A\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

So

$AP = PD$, where $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} = P$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Then $\det P = 1\cdot(2\cdot 1 - 1\cdot 3) = -1 \neq 0$ so $P^{-1}AP = D$.

Since
$$(A-2)\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ then } \begin{pmatrix}1\\0\\0\end{pmatrix}\in \ker(A-2)$$

Since
$$(A-2)\begin{pmatrix}0\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ then } \begin{pmatrix}0\\2\\1\end{pmatrix}\in \ker(A-2)$$

Since
$$(A-3)\begin{pmatrix}-3\\3\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \text{ then } \begin{pmatrix}0\\3\\1\end{pmatrix}\in \ker(A-3).$$

So
$$\ker(A-2)\neq 0 \quad \text{and} \quad \ker(A-3)\neq 0.$$

So $A-2$ is not invertible and $A-3$ is not invertible

So $\det(A-2)=0$ and $\det(A-3)=0$.

Let
$$A-t=\begin{pmatrix}2&-3&6\\0&5&-6\\0&1&0\end{pmatrix}-\begin{pmatrix}t&0&0\\0&t&0\\0&0&t\end{pmatrix}=\begin{pmatrix}2-t&-3&6\\0&5-t&-6\\0&1&0-t\end{pmatrix}.$$

Then
$$\begin{aligned}\det(A-t)&=(2-t)\big((5-t)(-t)+6\big)\\&=(2-t)(t^2-5t+6)\\&=(2-t)(t-2)(t-3)\\&=-(t-2)^2(t-3).\end{aligned}$$

Also

$\operatorname{tr}(A)=\operatorname{tr}(D)=7$ and $\det(A)=\det(D)=12=2\cdot 2\cdot 3$

Example 2 and 6 and 9

Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ and $A - t = \begin{pmatrix} 1-t & 4 \\ 1 & 1-t \end{pmatrix}$

Then

$$\det(A-t) = (1-t)^2 - 4 = 1 - 2t + t^2 - 4$$
$$= t^2 - 2t - 3 = (t-3)(t+1)$$

So $\det(A-3) = 0$ and $\det(A+1) = 0$.

So $\ker(A-3) \neq 0$ and $\ker(A+1) = 0$.

Then

$$\ker(A-3) = \ker\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \ker\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$$
$$= \mathrm{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \text{ since } \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

and

$$\ker(A+1) = \ker\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \ker\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$
$$= \mathrm{span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\} \text{ since } \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So

$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ since } (A-3)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ since } (A+1)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So

$$A\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

So

$$AP = PD \quad \text{where } P = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\text{and } D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

So $P^{-1}AP = D$.

Then $tr(A) = tr(D) = 3 - 1 = 2$ and

$$\det(A) = \det(D) = 3 \cdot (-1) = -3.$$

<u>Example 11</u> $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalisable

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $P^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

and

$$PAP^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{(ad-bc)}\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} -ca & a^2 \\ -c^2 & ca \end{pmatrix} \neq \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Since

$$\det(A - t) = \det\begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix} = t^2 = (t-0)^2$$

and

$$\ker(A - 0) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

A has only one linearly independent eigenvector.