A. Ram 30.08.2023
Linear Algebra 1

The <u>dimension of a vector space W</u> is the number of elements in a basis of W.

<u>Matrix monsters</u> Let $s,t \in \mathbb{Z}_{>0}$.
Let $A \in M_{s \times t}(\mathbb{R})$.

The <u>columnspace of A</u> is the span of the columns of A,

$$\mathrm{colspace}(A) = \left\{ c_1 \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + c_t \begin{pmatrix} | \\ a_t \\ | \end{pmatrix} \;\middle|\; c_1, \ldots, c_t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_t \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} \;\middle|\; c_1, \ldots, c_t \in \mathbb{R} \right\}$$

$$= \{ A\vec{c} \mid \vec{c} \in \mathbb{R}^t \} = \mathrm{im}(A)$$

The <u>solution space of A</u>, or <u>kernel of A</u>, is

$$\mathrm{solutionspace}(A) = \ker(A)$$
$$= \{ \vec{c} \in \mathbb{R}^s \mid A\vec{c} = 0 \}.$$

The <u>row space of A</u> is the span of the rows of A,

$$\mathrm{rowspace}(A) = \{ c_1 (- a_1 -) + \cdots + c_s (- a_s -) \mid c_1, \ldots, c_s \in \mathbb{R} \}$$

$$= \left\{ (c_1 \cdots c_s) \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_s & - \end{pmatrix} \;\middle|\; c_1, \ldots, c_s \in \mathbb{R} \right\}$$

$$= \{ (c_1 \cdots c_s) A \mid c_1, \ldots, c_s \in \mathbb{R} \}.$$

Examples 28 and 29 Let

$$S = \{\langle 1,3,-1,1\rangle, \langle 2,6,0,4\rangle, \langle 3,9,-2,4\rangle\}$$

Then
$$S = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ -2 \\ 4 \end{pmatrix} \right\}$$ and

$$\mathrm{span}(S) = \mathrm{colspace}(A) = \mathrm{im}(A), \text{ where}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}$$

Now
$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = PR$$

where $P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $A = PR$ and $R = P^{-1}A$ so that

(a) the rows of A are linear combinations of the rows of R

(b) the rows of R are linear combinations of the rows of A.

So $\mathrm{rowspace}(A) = \mathrm{rowspace}(R)$

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Linear Algebra 3

So

rowspace(A) has basis $\{(1\ 0\ 2), (0, 1\ {-\tfrac{1}{2}})\}$.

Next, colspace$(A) = \{P\vec{c} \mid \vec{c} \in$ colspace$(R)\}$

and colspace(R) has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

So colspace(A) has basis $\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix} \right\}$

Next, solutionspace(A) = solutionspace(R).

which is

$$\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \middle| \begin{matrix} c_1 + 2c_3 = 0 \\ c_2 - \tfrac{1}{2}c_3 = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \middle| \begin{matrix} c_1 = -2c_3 \\ c_2 = +\tfrac{1}{2}c_3 \\ c_3 = c_3 \end{matrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -2 \\ +\tfrac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

So ker(A) has basis $\left\{ \begin{pmatrix} -2 \\ +\tfrac{1}{2} \\ 1 \end{pmatrix} \right\}$

Example 31 Find a basis for the solution space of

$$x_1 + 2x_2 + x_3 + x_4 = 0$$
$$3x_1 + 5x_2 + 4x_3 + x_4 = 0.$$

Then

$$\text{solutionspace}\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

then

$$\ker\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \middle| \begin{array}{r} c_1 + 2c_2 + c_4 = 0 \\ c_3 - 2c_4 = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \middle| \begin{array}{l} c_1 = -2c_2 - c_4 \\ c_2 = c_2 \\ c_3 = -2c_4 \\ c_4 = c_4 \end{array} \right\}$$

$$= \left\{ c_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \middle| c_2, c_4 \in \mathbb{R} \right\}$$

So $\text{solutionspace}\begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix}$ has basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Example 32 Assume there exists $P \in M_{4\times 4}(\mathbb{R})$ such that P is invertible and

$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 2 & 0 & 1 & 0 \\ 5 & -3 & 7 & -6 \\ 1 & 1 & -1 & 3 \end{pmatrix} = P \cdot \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = PB$$

Since $A = PB$ and $B = P^{-1}A$ then

(a) The rows of A are linear combinations of the rows of B and

(b) The rows of B are linear combinations of the rows of A.

So rowspace(A) = rowspace(B)

So rowspace(A) has basis $\{(1\ -1\ 2\ -2), (0\ 2\ 3\ 4), (0\ 0\ 0\ 1)\}$

A basis of the column space is the first, second and $4^{\underline{th}}$ columns of A.

So

colspace(A) has basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -6 \\ 3 \end{pmatrix} \right\}$