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Linear Algebra
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Let V be an $\mathbb{R}$-vector space.
Let $W \subseteq V$ be a subspace of V.
Let $k \in \mathbb{Z}_{>0}$ and $S = \{w_1, \ldots, w_k\} \subseteq W$.
The set S is a basis of W if

(1) $\mathrm{span}(S) = W$

(2) S is linearly independent

<u>Example 2.3</u>

Let

$v_1 = (1,2,3)$

$v_2 = (3,6,9)$

$v_3 = (-1,0,-2)$

$v_4 = (1,4,4)$

Then

$v_2 = 3\cdot(1,2,3) = 3v_1$

$v_4 = (1,4,4) = 2(1,2,3) + (-1,0,-2)$

$= 2v_1 + v_3.$

Let $W = \mathrm{span}\{v_1, v_2, v_3, v_4\}$

$= \{c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$

$= \{c_1 v_1 + 3c_2 v_1 + c_3 v_3 + 2c_4 v_1 + c_4 v_3 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$

$= \{(c_1 + 3c_2 + 2c_4) v_1 + (c_3 + c_4) v_3 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$

$= \{d_1 v_1 + d_2 v_3 \mid d_1, d_2 \in \mathbb{R}\}$

$= \mathrm{span}\{v_1, v_3\}$

and v_3 is <u>not</u> a multiple of v_1.
So $\{v_1, v_3\}$ is a basis of W.

<u>Theorem</u> (in English)

(a) A basis of W is a maximal linearly independent subset of W.

(b) A basis of W is a minimal spanning subset of W.

(c) Any two bases of W have the same number of elements.

<u>Example 24</u> (and Example 16 and Topic 5 Examples 20 and 21).

Let
$$C = \{\langle 1,-1\rangle, \langle 2,4\rangle\}$$
$$B = \{\langle 1,0\rangle, \langle 0,1\rangle\} \quad \text{in } \mathbb{R}^2$$

Then
$$\langle 1,-1\rangle = 1\cdot\langle 1,0\rangle + (-1)\cdot\langle 0,1\rangle$$
$$\langle 2,4\rangle = 2\cdot\langle 1,0\rangle + 4\cdot\langle 0,1\rangle$$

The transition matrix from C to B is
$$P_{BC} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

Also
$$\langle 1,0\rangle = \tfrac{4}{6}\langle 1,-1\rangle + \tfrac{1}{6}\langle 2,4\rangle$$
$$\langle 0,1\rangle = \tfrac{-2}{6}\langle 1,-1\rangle + \tfrac{1}{6}\langle 2,4\rangle.$$

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The transition matrix from B to C

$$P_{CB} = \begin{pmatrix} \frac{4}{6} & \frac{-2}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

and $P_{CB}P_{BC} = 1$ and $P_{BC}P_{CB} = 1$.

Thus these matrices are square and B contains the same number of elements as C.

<u>Example 27</u> Let $W = P_2 = \text{span}\{1, x, x^2\}$,

$C = \{2+2x+5x^2, 1+x+x^2, 1+2x+3x^2\}$

$B = \{1, x, x^2\}$.

Then
$$2+2x+5x^2 = 2\cdot 1 + 2\cdot x + 5\cdot x^2$$
$$1+x+x^2 = 1\cdot 1 + 1\cdot x + 1\cdot x^2$$
$$1+2x+3x^2 = 1\cdot 1 + 2\cdot x + 3x^2$$

The transition matrix from C to B is

$$P_{BC} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix}$$

then

$$1 = \frac{1}{3}(2+2x+5x^2) + \frac{4}{3}(1+x+x^2) - (1+2x+3x^2)$$

$$x = \frac{-2}{3}(2+2x+5x^2) + \frac{1}{3}(1+x+x^2) + (1+2x+3x^2)$$

$$x^2 = \frac{1}{3}(2+2x+5x^2) - \frac{2}{3}(1+x+x^2)$$

The transition matrix from B to C is

$$P_{CB} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \\ -1 & 1 & 0 \end{pmatrix}$$

and $P_{CB}P_{BC} = 1$ and $P_{BC}P_{CB} = 1$.

Thus these matrices are square and the basis B contains the same number of elements as the basis C.