Linear Algebra
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Let V be an $\mathbb{R}$-vectorspace.

A subspace of V is a subset $W \subseteq V$ such that

(1) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

(2) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

(3) $0 \in W$

Let $k \in \mathbb{Z}_{>0}$ and $S = \{v_1, \ldots, v_k\}$ a subset of V.

Proposition span(S) is a subspace of V.

Proof span$(S) = \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$

To show: (1) If $w_1, w_2 \in$ span(S) then
$w + w_2 \in$ span(S)

(2) If $w \in$ span(S) and $c \in \mathbb{R}$ then
$cw \in$ span(S).

(1) Assume $w_1 = c_1 v_1 + \cdots + c_k v_k$
$w_2 = d_1 v_1 + \cdots + d_k v_k$ are in span(S)

Then $w_1 + w_2 = (c_1 + d_1) v_1 + \cdots + (c_k + d_k) v_k$ is in span(S) since $c_1 + d_1, \ldots, c_k + d_k \in \mathbb{R}$.

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(2) To show: If $w \in W$ and $c \in \mathbb{R}$ then $cw \in \mathrm{span}(S)$.

Assume $w = c_1 v_1 + \cdots + c_k v_k$ is in $\mathrm{span}(S)$ and $c \in \mathbb{R}$.

Then $cw = c(c_1 v_1 + \cdots + c_k v_k)$

$= cc_1 v_1 + \cdots + cc_k v_k$ is in $\mathrm{span}(S)$

since $cc_1, \ldots, cc_k \in \mathbb{R}$. //.

<u>Vector spaces</u> (think $V = M_{3\times 2}(\mathbb{R})$)

A <u>vector space</u> is a set V with functions

$$\begin{array}{ccc} V \times V \to V & \text{and} & \mathbb{R} \times V \to V \\ (v_1, v_2) \mapsto v_1 + v_2 & & (c, v) \mapsto cv \\ \text{addition} & & \text{scalar multiplication} \end{array}$$

such that

(A1) If $v_1, v_2, v_3 \in V$ then

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

(A2) If $v_1, v_2 \in V$ then $v_1 + v_2 = v_2 + v_1$

(A3) There exists $0 \in V$ such that if $v \in V$ then $0 + v = v$ and $v + 0 = v$.

(A4) If $v \in V$ then there exists $-v \in V$ such that

$$v + (-v) = 0 \quad \text{and} \quad (-v) + v = 0.$$

(S1) If $c_1, c_2 \in \mathbb{R}$ and $v \in V$ then

$$c_1(c_2 v) = (c_1 c_2) v.$$

(S2) If $v \in V$ then $1 \cdot v = v$.

(D1) If $c_1, c_2 \in \mathbb{R}$ and $v \in V$ then

$$(c_1 + c_2) v = c_1 v + c_2 v.$$

(D2) If $c \in \mathbb{R}$ and $v_1, v_2 \in V$ then

$$c(v_1 + v_2) = c v_1 + c v_2.$$

<u>Proposition</u> Let V be a vector space. Let $v \in V$

(a) $0 \cdot v = 0$

(b) $(-1) \cdot v = -v$.

<u>Proof</u> (a) $0 \cdot v = (0+0) \cdot v$ (definition of 0 in $\mathbb{R}$)

$= 0 \cdot v + 0 \cdot v$ (distributive law)

Add $-0 \cdot v$ to both sides:

$$0 \cdot v + (-0 \cdot v) = 0 \cdot v + 0 \cdot v + (-0 \cdot v)$$

So $\quad 0 = 0 \cdot v + 0 \quad$ (definition of $-0 \cdot v$)

So $\quad 0 = 0 \cdot v \quad$ (definition of 0 in V)

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(b) To show: $v + (-1)v = 0$.

$$v + (-1)\cdot v = 1\cdot v + (-1)\cdot v \quad \text{(action of } 1\in\mathbb{R} \text{ on } V)$$
$$= (1+(-1))\cdot v \quad \text{(distributive law)}$$
$$= 0\cdot v \quad \text{(definition of } -1 \text{ in } \mathbb{F})$$
$$= 0 \quad \text{(part (a))}.$$
//.

<u>Proposition</u> Let V be a vector space and let $W \subseteq V$. Assume W satisfies the conditions

(0) $W \neq \emptyset$

(1) If $w_1, w_2 \in W$ ~~and~~ then $w_1 + w_2 \in W$

(2) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Then W satisfies the conditions

(3) $0 \in W$

(4) If $w \in W$ then $-w \in W$.

<u>Proof</u> (3) To show: $0 \in W$.
Since $W \neq \emptyset$ there exists $w \in W$.
Since $0 = 0\cdot w$ then (2) gives $0 \in W$.

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(4) Assume $w \in W$.

To show: $-w \in W$.

Since $w \in W$ and $(-1) \in \mathbb{R}$ and (2) holds then $(-1) \cdot w \in W$.

Since $(-1) \cdot w = -w$ then $-w \in W$. $\square$.