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Linear Algebra
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PROOF MACHINE

Linearly independent sets.

Let V be an $\mathbb{R}$-vector space. Let $k \in \mathbb{Z}_{>0}$.
Let
$$S = \{v_1, v_2, \ldots, v_k\} \text{ be a subset of } V.$$

The set S is <u>linearly independent</u> if S satisfies:

if $c_1, \ldots, c_k \in \mathbb{R}$ and $c_1 v_1 + \cdots + c_k v_k = 0$

then $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

<u>Topic 4 Example 21</u> Let

$$S = \{1 + 2x + 5x^2,\ 1 + x + x^2,\ 1 + 2x + 3x^2\}$$

in $P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$.
Show that S is linearly independent.

<u>Topic 4 Example 22</u> Let

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

Show that S is linearly independent.

Example 22

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3\begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $c_1 = 0$ and $c_2 = 0$ and $c_3 = 0$.

Assume $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3\begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

then

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 0, \\ 3c_1 + c_2 + 10c_3 &= 0, \\ c_1 + c_2 + 4c_3 &= 0, \\ c_1 - c_2 + 2c_3 &= 0. \end{aligned}$$

So

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Use row reduction to find the solutions:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \text{ with } t \in \mathbb{R}.$$

So $c_1 = 0, c_2 = 0, c_3 = 0$ is not the only solution.

So S is <u>not</u> linearly independent.

Example 21

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1(1+2x+5x^2)+c_2(1+x+x^2)+c_3(1+2x+3x^2)=0$$

then $c_1=0$ and $c_2=0$ and $c_3=0$.

Assume $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1(1+2x+5x^2)+c_2(1+x+x^2)+c_3(1+2x+3x^2)=0.$$

Then

$$c_1+c_2+c_3=0,$$
$$2c_1+c_2+2c_3=0,$$
$$5c_1+c_2+3c_2=0.$$

So

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Use row reduction to find that this system has only one solution.

$$c_1=0,\ c_2=0,\ c_3=0.$$

So S is linearly independant.

Topic 4 Example 13 Show that

$$\langle 1,2,3\rangle \in \text{span}\{\langle 1,-1,2\rangle, \langle -1,1,2\rangle\}.$$

To show: There exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\langle 1,-1,2\rangle + c_2\langle -1,1,2\rangle = \langle 1,2,3\rangle.$$

To show: the system

$$\begin{aligned} c_1 - c_2 &= 1 \\ -c_1 + c_2 &= 2 \\ 2c_1 + 2c_2 &= 3 \end{aligned}$$

has a solution.

In matrix form the equations are

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Muliplying both sides by $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ gives

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Since the top row on the left side is all 0 and the top row on the right side is not 0 there do not exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\langle 1,-1,2\rangle + c_2\langle -1,1,2\rangle = \langle 1,2,3\rangle.$$

So $\langle 1,2,3\rangle \notin \text{span}\{\langle 1,-1,2\rangle, \langle -1,1,2\rangle\}$

Topic 4 Example 14 Show that

$$1-2x-x^2 \in \mathrm{span}\{1+x+x^2, 3+x^2\}.$$

To show: there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1+x+x^2)+c_2(3+x^2) = 1-2x-x^2.$$

To show: the system

$$\begin{aligned} c_1+3c_2 &= 1, \\ c_1+0c_2 &= -2, \\ c_1+c_2 &= -1, \end{aligned}$$

has a solution.

In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Skipping the row reduction steps,

$c_1 = -2$, $c_2 = 1$ is a solution.

So $-2(1+x+x^2)+1\cdot(3+x^2) = 1-2x-x^2$.

So $1-2x-x^2 \in \mathrm{span}\{1+x+x^2, 3+x^2\}$.