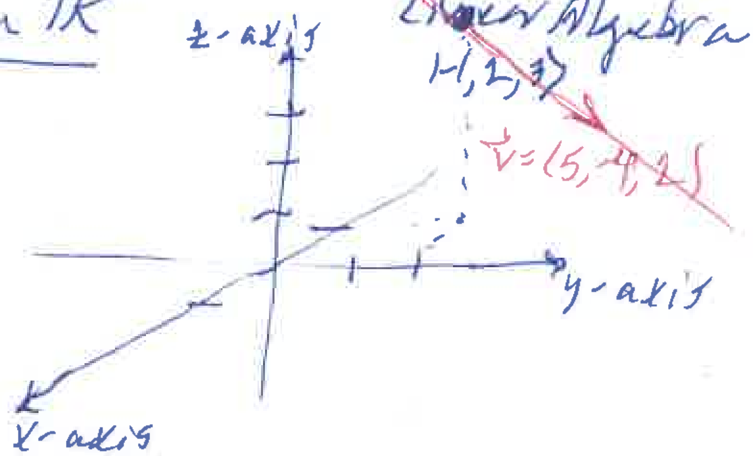
10.09.2023 ①
Linear Algebra

Points, lines and planes in $\mathbb{R}^3$



A point

$$x=-1$$
$$y=2$$
$$z=3$$

A line

$$x=-1+5t$$
$$y=2-4t \quad \text{with } t\in\mathbb{R}$$
$$z=3+2t$$

parametric form

The line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} \quad \text{with } t\in\mathbb{R}.$$

Vector form

or

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \operatorname{span}\left\{\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}\right\}$$

or

$$t = \frac{x+1}{5} = \frac{y-2}{-4} = \frac{z-3}{2}.$$

Cartesian form

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planes

$$x = -1 + s + 2t$$
$$y = 1 - s + 0t \quad \text{with } s,t \in \mathbb{R}.$$
$$z = 1 + 0s + t$$

parametric form

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{with } s,t \in \mathbb{R}$$

vector form

or

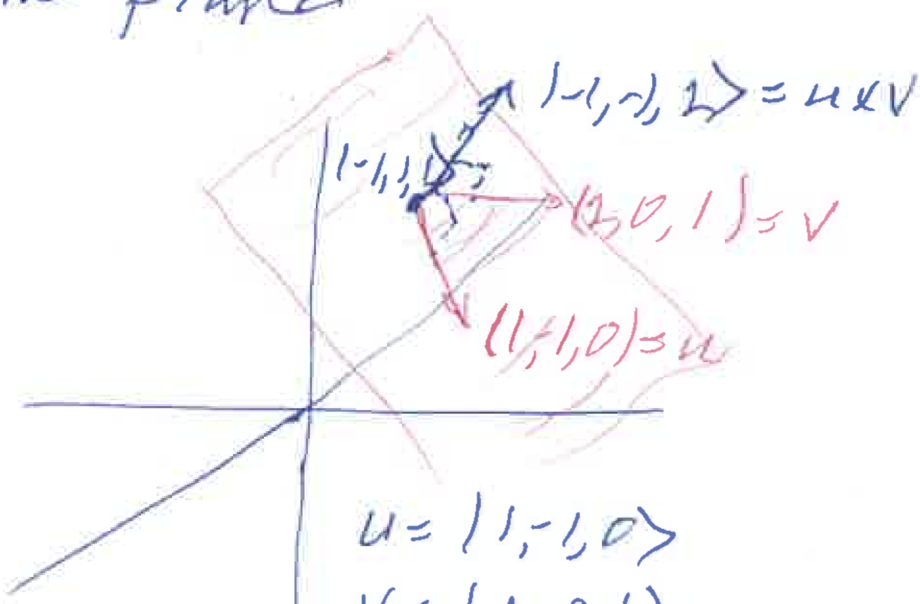
$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$$

or

$$-x - y + 2z = 2.$$

Cartesian form

The vector $\langle -1, -1, 2 \rangle$ is the normal vector to the plane



$u = \langle 1, -1, 0 \rangle$

$v = \langle 2, 0, 1 \rangle$

$$n = u \times v = \langle -1-0, -(1-0), 0-(-2) \rangle$$
$$= \langle -1, -1, 2 \rangle$$

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<u>Topic 3 Example 9</u> Find the vector equation of the line through the point $\langle 2,0,1\rangle$ which is parallel to the "enemy" line

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}.$$

The "enemy" line is

$$x = 1 + t$$
$$y = -2 - 2t$$
$$z = 6 + 2t$$

(since $t = \frac{x-1}{1}$, $t = \frac{y+2}{2}$, $t = \frac{z-6}{2}$).

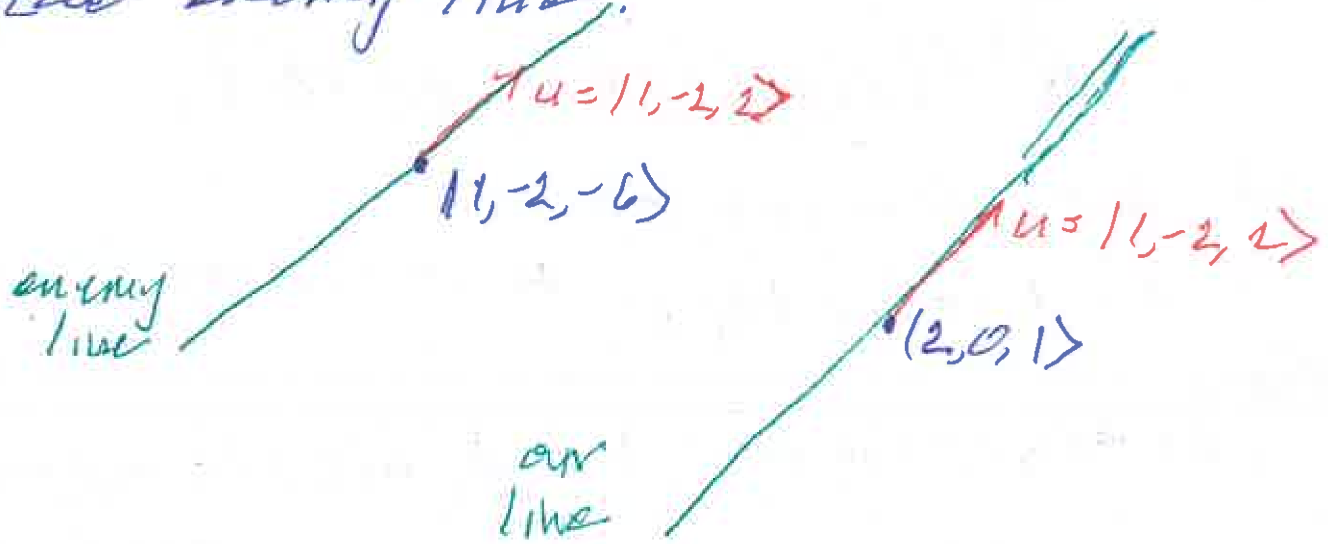
The line we want is

$$x = 2 + t$$
$$y = 0 - 2t$$
$$z = 1 + 2t$$

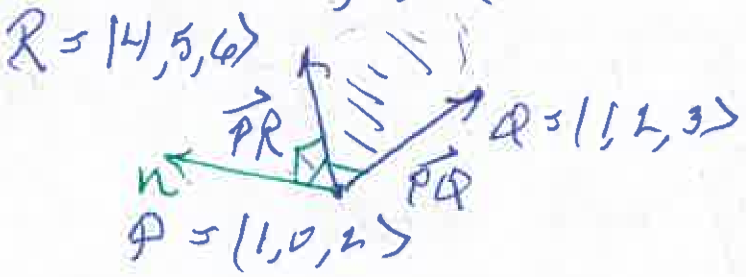
(equivalently, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ with $t \in \mathbb{R}$

or $\frac{x-2}{1} = \frac{y-0}{-2} = \frac{z-1}{2}$)

since this line goes through $\langle 2,0,1\rangle$ and has direction $\langle 1,-2,2\rangle$ and $\langle 1,-2,2\rangle$ is the same direction as the "enemy" line.



Topic 3 Example 11 Find the plane containing $P = \langle 1,0,2\rangle$, $Q = \langle 1,2,3\rangle$ and $R = \langle 4,5,6\rangle$.



$$u = \vec{PQ} = Q - P = \langle 1,2,3\rangle - \langle 1,0,2\rangle = \langle 0,2,1\rangle$$

$$v = \vec{PR} = R - P = \langle 4,5,6\rangle - \langle 1,0,2\rangle = \langle 3,5,4\rangle$$

$$n = u \times v = \langle 2\cdot 4 - 5\cdot 1, -(0\cdot 4 - 3\cdot 1), 0\cdot 5 - 3\cdot 2\rangle$$
$$= \langle 3, 3, -6\rangle.$$

$$\langle n | P\rangle = \langle 3,3,-6 | 1,0,2\rangle = 3 - 12 = -9.$$

So the plane is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \text{ with } s,t \in \mathbb{R}$$

or, equivalently,

$$3x + 3y - 6z = -9.$$

or, equivalently,

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \operatorname{span}\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}\right\}.$$

Topic 3 Example 12 Where does the line

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

intersect the plane $3x+2y+z=20$

The line is $t=\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ so the line is

$x=1+t$
$y=2+2t$ with $t\in\mathbb{R}$.
$z=3+3t$

A point on this line is in the plane $3x+2y+z=20$ if

$$3(1+t)+2(2+2t)+(3+3t)=20.$$

So $10+10t=20$. So $10t=10$. So $t=1$.

So
$x=1+1$
$y=2+2\cdot 1$ is on both the line and the plane.
$z=3+3\cdot 1$

So $(1+1, 2+2, 3+3)=(2,4,6)$ is the intersection point.

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Topic 3 Example 13 Find the line where $x+3y+2z=6$ and $3x+2y-z=11$ intersect.

In matrix form, the equations

$$\begin{matrix} 3x+2y-z=1 \\ x+3y+2z=6 \end{matrix} \quad \text{become} \quad \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

Multiply both sides by $y_1(3)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$

Multiply both sides by $d(1,-7)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{pmatrix}\begin{pmatrix} 6 \\ -7 \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

Multiply both sides by $x_{12}(3)^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

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So $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

So $x - z = 3$

$y + z = 1$.

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ with $z \in \mathbb{R}$.

So the line of intersection is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ with } t \in \mathbb{R}.$$