## What we covered last time

(1) The most amazing number system
size of matrices, addition, multiplication
scalar multiplication, inverses, identity, zero.

(2) Elementary matrices: Examples

$$d(5,1,4) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{Q}), \quad d(5,1,4)^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$x_{24}(6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in M_{5\times 5}(\mathbb{Q}), \quad x_{24}(6)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

## I owe you

(1) Topic 1 Example 6 and 7
(2) Topic 1 Example 10 and 11.

**Example** Solve $2x = 10$.
Multiply both sides by $2^{-1}$

$$2^{-1} \cdot 2x = 2^{-1} \cdot 10$$

So $x = 5$.

## Example 6

Solve
$$4x-2y+5z=31$$
$$2x-3y-2z=13$$
$$x-3y+2z=11$$

In matrix form this is

$$\begin{pmatrix}4 & -2 & 5\\ 2 & -3 & -2\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}31\\ 13\\ 11\end{pmatrix} \qquad (Eq)$$

For $c\in\mathbb{Q}$ let

$$y_1(c)^{-1}=\begin{pmatrix}0 & 1 & 0\\ 1 & -c & 0\\ 0 & 0 & 1\end{pmatrix} \text{ and } y_2(c)^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -c\end{pmatrix}.$$

These are the <u>row</u> <u>reducers</u>.

Multiply both sides of $(Eq)$ by $y_2(2)^{-1}$:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -2\end{pmatrix}\begin{pmatrix}4 & -2 & 5\\ 2 & -3 & -2\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -2\end{pmatrix}\begin{pmatrix}31\\ 13\\ 11\end{pmatrix}$$

So

$$(Eq1)\qquad \begin{pmatrix}4 & -2 & 5\\ 1 & -3 & 2\\ 0 & 3 & -6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}31\\ 11\\ -9\end{pmatrix}$$

i.e.
$R_2 \to R_3$
$R_3 \to R_2 - 2R_3$.

01.08.2013 (3)

Multipy both sides of (Eq1) by $y_1(4)^{-1}$: Linear Algebra A. Ram

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}$$

So

(Eq2) $$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}$$

i.e. $R_1 \to R_2$
$R_2 \to R_1 - 4R_3$.

Multiply both sides of (Eq2) by $y_2(\frac{10}{3})^{-1}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix}\begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}$$

So

(Eq3) $$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}$$

i.e. $R_2 \to R_3$
$R_3 \to R_2 - \frac{10}{3}R_3$.

Multiply both sides by $d(1,3,17)^{-1}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix}\begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}$$

So

(Eq4) $$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}$$

i.e. $R_1 \to R_1$
$R_2 \to \frac{1}{3}R_2$
$R_3 \to \frac{1}{17}R_3$

01.09.2013 (4)
Linear Algebra
A. Rao

Multiply both sides of (Eq 4) by $x_{12}(-3)^{-1}$:

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}$$

So (Eq 5)
$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
i.e. $R_1 \to R_1 + 3R_2$

Multiply both sides by $x_{13}(-4)^{-1}$:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

So (Eq 6)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$
i.e. $R_1 \to R_1 + 4R_3$

Multiply both sides of (Eq 6) by $x_{23}(-2)^{-1}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

So
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$
i.e. $R_2 \to R_2 + 2R_3$

So $x = 6$, $y = -1$, $z = 1$.

So there is only <u>one</u> <u>solution</u> in this case.

01.08.2023 (5)
Linear Algebra
A. Ram.

Let

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}$$

Our process gave

$$x_{23}(-2)^{-1} x_{13}(-4)^{-1} x_{12}(-3)^{-1} d(1,3,17)^{-1} y_2(\tfrac{10}{3})^{-1} y_1(4)^{-1} y_2(2)^{-1} \cdot A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

So

$$A^{-1} = x_{23}(-2)^{-1} x_{13}(-4)^{-1} x_{12}(-3)^{-1} d(1,3,17)^{-1} y_2(\tfrac{10}{3})^{-1} y_1(4)^{-1} y_2(2)^{-1}.$$

Then

$$A = y_2(2) y_1(4) y_2(\tfrac{10}{3}) d(1,3,17) x_{12}(-3) x_{13}(-4) x_{23}(-2)$$

(so that $A^{-1}A = 1$), where

$$y_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In other words,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{10}{3} & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

is a product of elementary matrices.