Linear Algebra Lecture 28.07.2023

What we covered last lecture

Linear Algebra.
A. Ram

(1) Where I'm from
Teaching and math experience
Research Activity

(2) My web page
My linear algebra page

(3) English/Math/Cartoon

(4) Converting equations to matrices and back.

(5) Converting English to math

(6) Usually zillions of solutions.
sometimes only one solution
sometimes no solutions.

I owe you

(a) Topic 1 Examples 6 and 7
(b) Topic 1 Examples 10 and 11
(c) Winston's question
(d) Address tutorial 1 features.

qantas Australian domestic flight network.

The most amazing number system is Linear Algebra
A. Ram

MATRICES

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{c} a, b \in \mathbb{Z} \\ b \neq 0 \end{array} \right\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

$\mathbb{R}$ = real numbers

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ with $i^2 = -1$.

Let $m, n \in \mathbb{Z}_{>0}$.

$$M_{m\times n}(\mathbb{Q}) = \left\{ \begin{array}{l} \text{matrices } A \text{ with} \\ \quad m \text{ rows} \\ \quad n \text{ columns} \\ \quad \text{entries in } \mathbb{Q} \end{array} \right\}$$

(1) Addition is entry by entry.

If the matrices A and B are not the same size then $A+B$ is not defined.

In math: Let $A \in M_{m\times n}(\mathbb{Q})$ and $B \in M_{r\times s}(\mathbb{Q})$.
If $m \neq r$ or $n \neq s$ then

$A + B$ is not defined.

28.07.2023 ③
Linear Algebra
A. Raur

(2) <u>Scalar multiplication</u>

(3) <u>Multiplication</u>: The (i,j) entry of AB is the i^{th} row of A "times" the j^{th} column of B.

Let $A \in M_{m \times n}(\mathbb{Q})$ and $B \in M_{r \times s}(\mathbb{Q})$.

If $n \neq r$ then AB is not defined

If $n = r$ then $AB \in M_{m \times s}(\mathbb{Q})$

row "times" column is illustrated by

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

(4) The <u>zero matrix</u> $O \in M_{m \times n}(\mathbb{Q})$.

<u>Example</u>

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{4 \times 2}(\mathbb{Q}).$$

(5) The <u>identity matrix</u> $1 \in M_{m\times m}(\mathbb{C})$. A. Ram

<u>Example:</u> $1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$

1 is a <u>square matrix</u>

<u>Winston's problem:</u>

Writing matrices in Latex is clumsy and the code is not pretty.

<u>Solution:</u> <u>ELEMENTARY MATRICES</u>.

(1) $E_{ij} \in M_{m\times n}(\mathbb{Q})$ has

1 in the (i,j) entry and

0 elsewhere.

<u>Example</u>

$E_{25} \in M_{3\times 7}(\mathbb{Q})$ is $E_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Then

$1 \in M_{3\times 3}(\mathbb{Q})$ is

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= E_{11} + E_{22} + E_{33}.$$

(2) $x_{ij}(c) \in M_{m\times m}(\mathbb{Q})$ is

$$x_{ij}(c) = 1 + cE_{ij}. \quad (\text{assume } i \neq j).$$

Example: $x_{24}(6) \in M_{5\times 5}(\mathbb{Q})$ is

$$x_{24}(6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$x_{24}(-6)\, x_{24}(6) = 1.$$

So

$$x_{24}(6)^{-1} = x_{24}(-6)$$

Let $A \in M_{m\times m}(\mathbb{Q})$. The <u>inverse of A</u> is $A^{-1} \in M_{m\times m}(\mathbb{Q})$ such that $A^{-1}A = 1$.

(3) $d(c_1, \ldots, c_m) \in M_{m\times m}(\mathbb{Q})$ is

$$d(c_1, \ldots, c_m) = c_1E_{11} + c_2E_{22} + \cdots + c_mE_{mm}.$$

Example:

$$d(5,1,4) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \text{ in } M_{3\times 3}(\mathbb{Q})$$

Since

$$\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ then}$$

$$d(\tfrac{1}{5}, 1, \tfrac{1}{4})\, d(5,1,4) = 1.$$

So $\quad d(5,1,4)^{-1} = d(\tfrac{1}{5}, 1, \tfrac{1}{4}).$

(4) $y_i(c) \in M_{m\times m}(\mathbb{Q})$ is

$$y_i(c) = 1 + (c-1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}$$

Example If $m = 4$ then

$$y_1(c) = \begin{pmatrix} c & 1 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \qquad y_2(c) = \begin{pmatrix} 1 & & & \\ & c & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

$$y_3(c) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & c & 1 \\ & & 1 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$y_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general,

$$y_i(c)^{-1} = 1 - E_{ii} + (-c-1)E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}.$$