10.10 Adjoints of linear transformations

Let V be an $\mathbb{F}$-vector space with a nondegenerate sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$. Let $f\colon V\to V$ be a linear transformation.

- The *adjoint of f with respect to $\langle,\rangle$* is the linear transformation $f^*\colon V\to V$ determined by
$$\text{if } x,y\in V \text{ then}\quad \langle f(x),y\rangle = \langle x, f^*(y)\rangle.$$
- The linear transformation f is *self adjoint* if f satisfies:
$$\text{if } x,y\in V\quad\text{then}\quad \langle f(x),y\rangle = \langle x, f(y)\rangle.$$
- The linear transformation f is an *isometry* if f satisfies:
$$\text{if } x,y\in V\quad\text{then}\quad \langle f(x),f(y)\rangle = \langle x, y\rangle.$$
- The linear transformation f is *normal* if $ff^*=f^*f$.

Let $\{w_1,\ldots,w_k\}$ be a basis of W and assume that the dual basis $\{w^1,\ldots,w^k\}$ of W exists. If $w=c_1w^1+\cdots c_kw^k$ then $c_j=\langle w,w_j\rangle$ and so
$$w=\langle w,w_1\rangle w^1+\cdots+\langle w,w_k\rangle w^k.$$
If $w\in W$ then
$$f^*(w)=\langle f^*(w),w_1\rangle w^1+\cdots+\langle f^*(w),w_k\rangle w^k=\langle w,f(w_1)\rangle w^1+\cdots+\langle w,f(w_k)\rangle w^k,$$
and this specifies $f^*\colon W\to W$ in terms of f. Then
$$f \text{ is self adjoint if } f=f^* \qquad\text{and}\qquad f \text{ is an isometry if } ff^*=1,$$

HW: Let $V=\mathbb{F}^n$ with basis $(e_1,\ldots,e_n)$ and inner product given by
$$e_i=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\quad\text{with 1 in the ith row}\quad\text{and}\quad \langle e_i,e_j\rangle=\delta_{ij}.$$
Let $f\colon V\to V$ be a linear transformation of V and let A be the matrix of f with respect to the basis $(e_1,\ldots,e_n)$. Show that, with respect to the basis $(e_1,\ldots,e_n)$,
$$\text{the matrix of } f^* \text{ is}\qquad A^*=\overline{A}^t.$$
Since
$$\sum_{i=1}^n A^*(i,j)e_i=f^*(e_j)=\sum_{i=1}^n\langle e_j,f(e_i)\rangle e_i=\sum_{i=1}^n\sum_{k=1}^n\langle e_j,A(k,i)e_k\rangle e_i=\sum_{i=1}^n\overline{A(j,i)}e_i,$$
then $A^*(i,j)=\overline{A(j,i)}$.