10.11 The Spectral theorem

Let $A \in M_n(\mathbb{C})$ and let $V = \mathbb{C}^n$ with inner product given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1\overline{y_1} + \cdots x_n\overline{y_n}. \tag{10.1}$$

Let $A \in M_n(\mathbb{C})$.

- The *adjoint of* A is the matrix $A^* \in M_n(\mathbb{C})$ given by $A^*(i,j) = \overline{A(j,i)}$.
- The matrix A is *self adjoint* if $A = A^*$.
- The matrix A is *unitary* if $AA^* = 1$.
- The matrix A is *normal* if $AA^* = A^*A$.

Write $A^* = \overline{A}^t$. The *unitary group* is

$$U_n(\mathbb{C}) = \{U \in M_n(\mathbb{C}) \mid UU^* = 1\}.$$

Theorem 10.10. *Let $V = \mathbb{C}^n$ with inner product given by (17.1). The function*

$$\begin{matrix} \left\{ \begin{matrix} \text{ordered orthonormal bases} \\ (u_1, \ldots, u_n) \text{ of } \mathbb{C}^n \end{matrix} \right\} & \longrightarrow & U_n(\mathbb{C}) \\ (u_1, \ldots, u_n) & \longmapsto & U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \end{matrix} \quad \text{is a bijection.}$$

The following proposition explains the role of normal matrices.

Proposition 10.11. *Let $V = \mathbb{C}^n$ with inner product given by (17.1). Let*

$$A \in M_n(\mathbb{C}), \qquad \lambda \in \mathbb{C} \qquad \text{and} \qquad V_\lambda = \ker(\lambda - A).$$

If $AA^* = A^*A$ then

$$V_\lambda \text{ is } A\text{-invariant}, \quad V_\lambda^\perp \text{ is } A\text{-invariant}, \quad V_\lambda \text{ is } A^*\text{-invariant and} \quad V_\lambda^\perp \text{ is } A^*\text{-invariant.}$$

Theorem 10.12. *(Spectral theorem)*
Let $n \in \mathbb{Z}_{>0}$ and $V = \mathbb{C}^n$ with inner product given by (17.1).

(a) Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$. Then there exists a unitary $U \in M_n(\mathbb{C})$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$U^{-1}AU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

(b) Let $f: V \to V$ be a linear transformation such that $ff^* = f^*f$. Then there exists an orthonormal basis $(u_1, \ldots, u_n)$ of V consisting of eigenvectors of f.

HW: Show that if $A \in M_n(\mathbb{C})$ is self adjoint then its eigenvalues are real.

HW: Show that if $U \in M_n(\mathbb{C})$ is unitary then its eigenvalues have absolute value 1.