10.9 Orthonormal bases

10.9.1 Orthonormal sequences

A *Hermitian form* is a sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$ such that

(H) If $v,w\in V$ then $\langle v,w\rangle=\overline{\langle w,v\rangle}$.

An *orthonormal sequence* in V is a sequence $(b_1,b_2,\ldots)$ in V such that

$$\text{if } i,j\in\mathbb{Z}_{>0} \quad \text{then} \quad \langle b_i,b_j\rangle=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Proposition 10.8. *Let V be an $\mathbb{F}$-vector space with a Hermitian form. An orthonormal sequence $(a_1,a_2,\ldots)$ in V is linearly independent.*

10.9.2 Gram-Schmidt

Let $n\in\mathbb{Z}_{>0}$ and let V be an inner product space with $\dim(V)=n$. An *orthonormal basis of V*, or *self-dual basis of V*, is a basis $\{u_1,\ldots,u_n\}$ such that

$$\text{if } i,j\in\{1,\ldots,n\} \quad \text{then} \quad \langle u_i,u_j\rangle=\begin{cases}0, & \text{if } i\neq j,\\ 1, & \text{if } i=j.\end{cases}$$

An *orthogonal basis in V* is a basis $\{b_1,\ldots,b_n\}$ such that

$$\text{if } i,j\in\{1,\ldots,n\} \quad \text{and } i\neq j \quad \text{then} \quad \langle b_i,b_j\rangle=0.$$

The following theorem guarantees that, in some favourite examples, orthonormal bases exist.

Theorem 10.9. *(Gram-Schmidt) Let V be an $\mathbb{F}$-vector space with a sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$. Assume that $\langle,\rangle$ is nonisotropic and that $\langle,\rangle$ is Hermitian i.e.,*

(1) (Nonisotropy condition) If $v\in V$ and $\langle v,v\rangle=0$ then $v=0$, and

(2) (Hermitian condition) If $v_1,v_2\in V$ then $\langle v_2,v_1\rangle=\overline{\langle v_1,v_2\rangle}$.

Let $p_1,p_2,\ldots$ be a sequence of linear independent elements of V.

(a) Define $b_1=p_1$ and

$$b_{n+1}=p_{n+1}-\frac{\langle p_{n+1},b_1\rangle}{\langle b_1,b_1\rangle}b_1-\cdots-\frac{\langle p_{n+1},b_n\rangle}{\langle b_n,b_n\rangle}b_n, \qquad \text{for } n\in\mathbb{Z}_{>0}.$$

Then $(b_1,b_2,\ldots)$ is an orthogonal sequence in V.

(b) Assume that $\mathbb{F}$ is a field in which square roots can be made sense of and that if $v\in V$ and $v\neq 0$ then $\langle v,v\rangle\neq 0$. Define

$$\|v\|=\sqrt{\langle v,v\rangle}, \qquad \text{for } v\in V.$$

Let $(b_1,\ldots,b_n)$ be an orthogonal basis of V. Define

$$u_1=\frac{b_1}{\|b_1\|}, \quad \ldots\;, \quad u_n=\frac{b_n}{\|b_n\|}.$$

Then $(u_1,\ldots,u_n)$ is an orthonormal basis of V.