10 Bilinear forms and quadratic forms

10.1 Bilinear forms

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. A *bilinear form on* V is a function

$$\begin{array}{rccc} \langle,\rangle\colon & V\times V & \to & \mathbb{F} \\ & (v,w) & \longmapsto & \langle v,w\rangle \end{array} \qquad \text{such that}$$

(a) If $v_1, v_2, w \in V$ then $\langle v_1+v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$,

(b) If $v, w_1, w_2 \in V$ then $\langle v, w_1+w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$,

(c) If $c\in\mathbb{F}$ and $v,w\in V$ then $\langle cv, w\rangle = c\langle v,w\rangle$,

(d) If $c\in\mathbb{F}$ and $v,w\in V$ then $\langle v, cw\rangle = c\langle v,w\rangle$.

A bilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$ is *symmetric* if it satsfies:

(S) If $v,w\in V$ then $\langle v,w\rangle = \langle w,v\rangle$.

A bilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$ is *skew-symmetric* if it satsfies:

(A) If $v,w\in V$ then $\langle v,w\rangle = -\langle w,v\rangle$.

10.2 Sesquilinear forms

Let $\mathbb{F}$ be a field and let $\overline{}\colon \mathbb{F}\to\mathbb{F}$ be a function that satisfies:

$$\text{if } c, c_1, c_2 \in \mathbb{F} \text{ then } \quad \overline{c_1+c_2} = \overline{c_1}+\overline{c_2}, \qquad \overline{c_1c_2} = \overline{c_2}\,\overline{c_1} \quad \text{and} \quad \overline{1}=1 \quad \text{and} \quad \overline{\overline{c}} = c.$$

The favourite example of such a function is complex conjugation. The other favourite example is the identity map $\mathrm{id}_{\mathbb{F}}$.

Let V be an $\mathbb{F}$-vector space. A *sesquilinear form on* V is a function

$$\begin{array}{rccc} \langle,\rangle\colon & V\times V & \to & \mathbb{F} \\ & (v,w) & \longmapsto & \langle v,w\rangle \end{array} \qquad \text{such that}$$

(a) If $v_1, v_2, w \in V$ then $\langle v_1+v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$,

(b) If $v, w_1, w_2 \in V$ then $\langle v, w_1+w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$,

(c) If $c\in\mathbb{F}$ and $v,w\in V$ then $\langle cv, w\rangle = c\langle v,w\rangle$,

(d) If $c\in\mathbb{F}$ and $v,w\in V$ then $\langle v, cw\rangle = \overline{c}\langle v,w\rangle$.

A *Hermitian form* is a sesquilinear form $\langle,\rangle\colon V\times V\to\mathbb{F}$ such that

(H) If $v,w\in V$ then $\langle v,w\rangle = \overline{\langle w,v\rangle}$.