Topic 4. Example 7. Is $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ a subspace of $\mathbb{R}^3$?

A *subspace of* $\mathbb{R}^3$ is a subset $W \subseteq \mathbb{R}^3$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof.

(a) Assume $w_1 = |a, b, c\rangle \in W$ and $w_2 = |x, y, z\rangle \in W$.
Then $a + b + c = 0$ and $x + y + z = 0$.
Then $w_1+w_2 = |a+x, b+y, c+z\rangle$ and $(a+x)+(b+y)+(c+z) = (a+b+c)+(x+y+z) = 0+0 = 0$.
So $w_1 + w_2 \in W$.
(b) $0 = |0, 0, 0\rangle$ satisfies $0 + 0 + 0 = 0$. So $0 \in W$.
(c) Assume $w = |x, y, z\rangle \in W$.
Then $x + y + z = 0$.
Then $-w = |-x, -y, -z\rangle$ and $(-x) + (-y) + (-z) = -(x + y + z) = -0 = 0$.
So $-w \in W$.
(d) Assume $w = |x, y, z\rangle \in W$ and $c \in \mathbb{R}$.
Then $x + y + z = 0$.
Then $cw = |cx, cy, cz\rangle$ and $cx + cy + cz = c(x + y + z) = c \cdot 0 = 0$.
So $cw \in W$.

So W is a subspace of $\mathbb{R}^3$. □

Topic 4. Example 8. Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?

A *subspace of* $\mathbb{R}^2$ is a subset $L \subseteq \mathbb{R}^2$ such that

(a) If $w_1, w_2 \in L$ then $w_1 + w_2 \in L$,
(b) $0 \in L$,
(c) If $w \in L$ then $-w \in L$,
(d) If $w \in L$ and $c \in \mathbb{R}$ then $cw \in L$.

Since $0 = |0, 0\rangle$ and $0 \neq 2 \cdot 0 + 1$ then $0 \notin L$.
So L is not a subspace of $\mathbb{R}^2$.

Topic 4. Example ??. Is $W = \{|x, y\rangle \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ a subspace of $\mathbb{R}^2$?

A *subspace of* $\mathbb{R}^2$ is a subset $W \subseteq \mathbb{R}^2$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Let $w = |1, 1\rangle$. Since $1 \geq 0$ and $1 \geq 0$ then $w \in W$.
Let $c = -1 \in \mathbb{R}$.
Then $cw = (-1) \cdot |1, 1\rangle = |-1, -1\rangle$ and $-1 \not\geq 0$. So $cw \notin W$.
So W is not a subspace of $\mathbb{R}^2$.