Topic 4. Example 20. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2,0,0), (6,1,7), (2,-1,2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$ then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} 2c_1 + 6c_2 + 2c_3 &= 0, \\ c_2 - c_3 &= 0, \\ 7c_2 + 2c_3 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & -1 \\ 0 & 7 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(Ex20)}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has only one solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent.

Topic 4. Example 22. Let S be the subset of $M_2(\mathbb{F})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Then

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 0, \\ 3c_1 + c_2 + 10c_3 &= 0, \\ c_1 + c_2 + 4c_3 &= 0, \\ c_1 - c_2 + 2c_3 &= 0, \end{aligned} \qquad \text{or, equivalently,} \qquad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(Ex22)}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ is not the only solution.
So S is linearly independent.

Here is a check that $c_1 = -3$, $c_2 = 1$, $c_3 = -1$ is a solution:

$$-3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = -3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -9 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$