Topic 4. Example 16. Let S be the subset of $\mathbb{R}^2$ given by

$$S = \{|1,-1\rangle, |2,4\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^2.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$
(b) $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar mutliplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$.

(b) To show: $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,
To show: $\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$c_1|1,-1\rangle + c_2|2,4\rangle = |1,0\rangle \qquad \text{and} \qquad d_1|1,-1\rangle + d_2|2,4\rangle = |0,1\rangle.$$

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{then} \qquad \begin{matrix} \frac{2}{3}|1,-1\rangle + \frac{1}{6}|2,4\rangle = |1,0\rangle, \text{ and} \\ -\frac{1}{3}|1,-1\rangle + \frac{1}{6}|2,4\rangle = |0,1\rangle. \end{matrix}$$

So $|1,0\rangle \in \mathbb{R}\text{-span}(S)$ and $|0,1\rangle \in \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$.