Topic 4. Example 10. Is the set of real 2×2 matrices whose trace is equal to 0 a subspace of $M_{2\times 2}(\mathbb{R})$?

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $W \subseteq M_{2\times 2}(\mathbb{R})$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,

(b) $0 \in W$,

(c) If $w \in W$ then $-w \in W$,

(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof. The set of real 2×2 matrices whose trace is equal to 0 is

$$W = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ \Big|\ a_{11} + a_{22} = 0 \right\}.$$

(a) Assume $w_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $w_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in W$.
Then $a_{11} + a_{22} = 0$ and $b_{11} + b_{22} = 0$.
Then $w_1 + w_2 = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$ and

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0.$$

So $w_1 + w_2 \in W$.

(b) $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $(0, 0, 0)$ satisfies $0 + 0 = 0$. So $0 \in W$.

(c) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$.
Then $a_{11} + a_{22} = 0$.
Then $-w = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$ and $(-a_{11}) + (-a_{22}) = -(a_{11} + a_{22}) = -0 = 0$.
So $-w \in W$.

(d) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $c \in \mathbb{R}$.
Then $a_{11} + a_{22} = 0$.
Then $cw = c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$ and

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = c \cdot 0 = 0.$$

So $cw \in W$.

So W is a subspace of $M_{2\times 2}(\mathbb{R})$. □