Topic 3. Example 1. If $p = |1,3,1,2\rangle$ and $q = |2,1,-1,3\rangle$ in $\mathbb{R}^4$ then $p - q = |1,-2,0,1\rangle$ and

$$\|p-q\| = \|\,|1,-2,0,1\rangle\| = \sqrt{1^2+(-2)^2+0^2+1^2} = \sqrt{1+4+0+1} = \sqrt{6}.$$

Topic 3. Example 2. If $u = |0,2,2,-1\rangle$ and $v = |-1,1,1,-1\rangle$ in $\mathbb{R}^4$ then

$$\langle u|v\rangle = \langle 0,2,2,-1|-1,1,1,-1\rangle = 0\cdot(-1)+2\cdot 1+2\cdot 1+(-1)\cdot(-1) = 0+2+2+1 = 5$$

and

$$\|u\| = \sqrt{0+4+4+1} = \sqrt{9} = 3 \qquad \text{and} \qquad \|v\| = \sqrt{1+1+1+1} = \sqrt{4} = 2.$$

Since $|5| \le 3\cdot 2$ we observe that, in this case,

$$|\langle u|v\rangle| \le \|u\|\cdot\|v\|.$$

Topic 3. Example 4. Let $u = |2,-1,-2\rangle$ and $v = |2,1,3\rangle$. Find vectors v_1 and v_2 such that $v = v_1 + v_2$ and v_1 is parallel to u and v_2 is perpendicular to u.
Since the projection of v onto u is parallel to u then let

$$v_1 = \mathrm{proj}_u(v) = \frac{1}{\|u\|^2}\langle u|v\rangle u = \frac{1}{9}\cdot(-3)\cdot u = \tfrac{-1}{3}\,|2,-1,-2\rangle = \left|\tfrac{-2}{3},\tfrac{1}{3},\tfrac{2}{3}\right\rangle$$

and

$$v_2 = u - v_1 = |2,-1,-2\rangle - \left|\tfrac{-2}{3},\tfrac{1}{3},\tfrac{2}{3}\right\rangle = \left|\tfrac{8}{3},\tfrac{2}{3},\tfrac{7}{3}\right\rangle.$$

Then $u = v_1 + v_2$ and v_1 is parallel to u and v_2 is perpendicular to u.