2.3 Some examples

2.3.1 Matrix addition

1. (addition of matrices is componentwise, entry by entry)

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 3+1 & 2+6 & 2+3 & 1+4 \\ 0+5 & 7+6 & 7-9 & 5+8 \\ 1-5 & 3.2-2 & 5+i\sqrt{2} & -1+0 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 8 & 5 & 5 \\ 5 & 13 & -2 & 13 \\ -4 & 1.2 & 5+i\sqrt{2} & -1 \end{pmatrix}$$

2. (addition of matrices of different sizes is not defined)

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix} \qquad \text{is not defined.}$$

3. (The meaning of the 0 matrix)

$$0 + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix}$$

4. (the meaning of $-A$)

$$-\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -6 & -2 & -1 \\ 0 & -7 & 9 & -5 \\ -1 & -3.2 & -5 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

2.3.2 Scalar multiplication

1. (scalar multiplication by c multiplies each entry by c)

$$2i \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 3.2 & i\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 2i\cdot 1 & 2i\cdot 2 & 2i\cdot 3 & 2i\cdot 4 \\ 2i\cdot 5 & 2i\cdot 6 & 2i\cdot 7 & 2i\cdot 2 \\ 2i\cdot(-5) & 2i\cdot 3.2 & 2i\cdot i\sqrt{2} & 2i\cdot 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2i & 4i & 6i & 8i \\ 10i & 12i & 14i & 4i \\ -10i & 6.4i & -2\sqrt{2} & 0 \end{pmatrix}$$

2.3.3 Products

1. (a 1×4 matrix multiplied by a 4×1 matrix is a 1×1 matrix)

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix} = (4 \cdot (-1) + 2 \cdot 2 + 6 \cdot 0 + 3 \cdot (-2i)) = (-4 + 4 + 0 - 6i) = (-6i).$$

2. (multiplication is not defined unless the number of columns of the first matrix is the same as the number of rows of the second matrix)

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \end{pmatrix} \quad \text{is not defined.}$$

3. (the inner product of two vectors)

$$\begin{aligned} \langle (4,2,6,3)^t, (-1,2,0,-2i)^t \rangle &= \begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix} \\ &= (4 \cdot (-1) + 2 \cdot 2 + 6 \cdot 0 + 3 \cdot (-2i)) = (-4 + 4 + 0 - 6i) = (-6i). \end{aligned}$$

4. Using that $i^2 = -1$,

$$\begin{aligned} &\begin{pmatrix} 4 & 2 & 6 & 3 \\ 1 & 2 & -3 & 2i \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ -2i & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 \cdot (-1) + 2 \cdot 2 + 6 \cdot 0 + 3 \cdot (-2i) & 4 \cdot 0 + 2 \cdot 0 + 6 \cdot 0 + 3 \cdot 0 & 4 \cdot 1 + 2 \cdot 1 + 6 \cdot 1 + 3 \cdot 1 \\ 1 \cdot (-1) + 2 \cdot 2 + (-3) \cdot 0 + 2i \cdot (-2i) & 1 \cdot 0 + 2 \cdot 0 + (-3) \cdot 0 + 2i \cdot 0 & 1 \cdot 1 + 2 \cdot 1 + (-3) \cdot 1 + 2i \end{pmatrix} \\ &= \begin{pmatrix} -6i & 0 & 15 \\ 7 & 0 & 2i \end{pmatrix} \end{aligned}$$

5. Using $i^2 = -1$,

$$\begin{aligned} \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 6 & -3 \\ 3 & 2i \end{pmatrix} &= \begin{pmatrix} -1 \cdot 4 + 2 \cdot 2 + 0 \cdot 6 - 2i \cdot 3 & -1 \cdot 1 + 2 \cdot 2 + 0 \cdot (-3) - 2i \cdot 2i \\ 0 \cdot 4 + 0 \cdot 2 + 0 \cdot 6 + 0 \cdot 3 & 0 \cdot 1 + 0 \cdot 2 + 0 \cdot (-3) + 0 \cdot 2i \\ 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 6 + 1 \cdot 3 & 1 \cdot 1 + 1 \cdot 2 + 1 \cdot (-3) + 1 \cdot 2i \end{pmatrix} \\ &= \begin{pmatrix} -6i & 7 \\ 0 & 0 \\ 15 & 2i \end{pmatrix} \end{aligned}$$

6. (the identity matrix 1)

$$1 \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot 1 = \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

7. (the inverse of the 5×5 matrix $y_3(c)$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

8. (the inverse of a diagonal matrix)

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1,$$

verifying that $d(1, \frac{1}{2}, \frac{7}{8}, 1, 5)d(\frac{1}{4}, 2\frac{8}{7}, 1, \frac{1}{5}) = 1$ and $d(\frac{1}{4}, 2\frac{8}{7}, 1, \frac{1}{5})d(1, \frac{1}{2}, \frac{7}{8}, 1, 5) = 1$

9. (the inverse of the 5×5 matrix $x_{14}(7)$)

$$\begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1,$$

verifying that $x_{14}(7)x_{14}(-7) = 1$ and $x_{14}(-7) = x_{14}(7) = 1$.

10. (the inverse of the 5×5matrix s_{23})

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1,$$

verifying that $s_{23}^2 = 1$.