Topic 1. Example 7. Let us solve the equation

$$\begin{pmatrix}1&0&1\\0&2&2\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-2\\4\\-3\end{pmatrix}.$$

Let

$$d(1,2,1)^{-1}=\begin{pmatrix}1^{-1}&0&0\\0&2^{-1}&0\\0&0&1^{-1}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{pmatrix}.$$

Multiplying both sides of the equation by $d(1,2,1)^{-1}$ to get

$$\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-2\\2\\-3\end{pmatrix}\qquad\text{which gives}\qquad\begin{array}{l}x_1=-2+x_3,\\x_2=2-x_3,\\0x_3=-3.\end{array}$$

So

$$\mathrm{Sol}(Ax=b)=\emptyset.$$

Let

$$x_{23}(c)=\begin{pmatrix}1&0&0\\0&1&c\\0&0&1\end{pmatrix},\qquad x_{13}(c)=\begin{pmatrix}1&0&c\\0&1&0\\0&0&1\end{pmatrix}\qquad\text{and}\qquad 1_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

Then

$$\begin{pmatrix}1&0&1\\0&2&2\\0&0&0\end{pmatrix}=d(1,2,1)\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}=d(1,2,1)\cdot 1_2\cdot x_{23}(1)x_{13}(1)$$

So that if

$$A=\begin{pmatrix}1&0&1\\0&2&2\\0&0&0\end{pmatrix}\qquad\text{then}\qquad A=P\cdot 1_2\cdot Q\quad\text{with}\quad P=d(1,2,1)\quad\text{and}\quad Q=x_{23}(1)x_{13}(1).$$

Topic 1. Example 8.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2\\-4\\15\end{pmatrix}\qquad\text{which gives}\qquad\begin{array}{l}x_1=2,\\x_2=-4,\\x_3=15.\end{array}$$

So

$$\mathrm{Sol}(Ax=b)=\left\{\begin{pmatrix}2\\-4\\15\end{pmatrix}\right\}.$$