Topic 1. Example 6 (and 5). Let us solve the equation

$$\begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Let

$$y_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}, \qquad 1_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix},$$

so that $x_{ij}(c)^{-1}$ has $-c$ in the (i,j) entry, 1 on the diagonal and 0 everywhere else. Let

$$d(c_1, c_2, c_3)^{-1} = \begin{pmatrix} c_1^{-1} & 0 & 0 \\ 0 & c_2^{-1} & 0 \\ 0 & 0 & c_3^{-1} \end{pmatrix} \qquad \text{be the diagonal matrix with } c_1^{-1}, c_2^{-1}, c_3^{-1}$$

on the diagonal.

Multiply both sides of the equation by $y_2(2)^{-1}$ to get

$$\begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}$$

Multiply both sides by $y_1(4)^{-1}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}$$

Multiply both sides by $y_2(\frac{10}{3})^{-1}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}$$

Multiply both sides by $\mathrm{diag}(1, 3, -17)^{-1}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}$$

Multiply both sides by $x_{12}(3)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Multiply both sides by $x_{13}(-4)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

Multiply both sides by $x_{23}(-2)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}, \qquad \text{which gives} \qquad \begin{matrix} x = 6, \\ y = -1, \\ z = 1. \end{matrix}$$

So

$$\mathrm{Sol}\left(\begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Remark 10.2. The process of solving the equations has computed

$$P^{-1} \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}.$$

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