Topic 1. Example 11. Let us solve the equation

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 3 & -6 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ b \end{pmatrix}.$$

Let

$$y_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad y_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}.$$

Multiply both sides of the equation by $y_2(\frac{2}{3})^{-1}$ to get

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ b \\ 7-\frac{2}{3}b \end{pmatrix}.$$

Multiply both sides by $y_1(\frac{1}{3})^{-1}$ to get

$$\begin{pmatrix} 3 & -6 & a \\ 0 & 0 & 1-\frac{1}{3}a \\ 0 & 1 & 1-\frac{2}{3}a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 4-\frac{1}{3}b \\ 7-\frac{2}{3}b \end{pmatrix}.$$

Multiply both sides by $y_2(0)^{-1}$ to get

$$\begin{pmatrix} 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \\ 0 & 0 & 1-\frac{1}{3}a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7-\frac{2}{3}b \\ 4-\frac{1}{3}b \end{pmatrix}.$$

Case 1: If $1-\frac{1}{3}a = 0$ and $4-\frac{1}{3}b \neq 0$ then

$$\begin{pmatrix} 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7-\frac{2}{3}b \\ 4-\frac{1}{3}b \end{pmatrix} \qquad \text{gives} \qquad 0x+0y+0z = 4-\frac{1}{3}b \neq 0,$$

and

$$\mathrm{Sol}\left(\begin{pmatrix} 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7-\frac{2}{3}b \\ 4-\frac{1}{3}b \end{pmatrix}\right) = \emptyset.$$

Case 2: If $1-\frac{1}{3}a = 0$ and $4-\frac{1}{3}b = 0$ then

$$\begin{pmatrix} 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7-\frac{2}{3}b \\ 0 \end{pmatrix} \qquad \text{gives} \qquad \begin{array}{l} y = (7-\frac{2}{3}b) - (1-\frac{2}{3}a)z, \\ x = \frac{b}{3} + 2y - \frac{a}{3}z, \\ \text{no restriction on } z. \end{array}$$

Since $x = \frac{b}{3} + 2y - \frac{a}{3}z = \frac{b}{3} + 14 - \frac{4}{3}b - (2-\frac{4}{3})z - \frac{a}{3}z = (14-b) - z$ then

$$\mathrm{Sol}\left(\begin{pmatrix} 3 & -6 & a \\ 0 & 1 & 1-\frac{2}{3}a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7-\frac{2}{3}b \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 14-b \\ 7-\frac{2}{3}b \\ 0 \end{pmatrix} + \mathrm{span}\left\{\begin{pmatrix} -1 \\ -(1-\frac{2}{3}a) \\ 1 \end{pmatrix}\right\}.$$

Case 3: If $1-\frac{1}{3}a = 0$ and $4-\frac{1}{3}b = 0$ then there is a unique solution.

In summary,

If $1-\frac{1}{3}a=0$ and $4-\frac{1}{3}b\neq 0$ then there are no solutions.
If $1-\frac{1}{3}a=0$ and $4-\frac{1}{3}b=0$ then there is no restriction on z.
If $1-\frac{1}{3}a\neq 0$ then there is a unique solution.

Alternatively,

If $a=3$ and $b\neq 12$ then there are no solutions.
If $a=3$ and $b=12$ then there is no restriction on z.
If $a\neq 3$ then there is a unique solution.

Normal form:

$$\begin{aligned}
\begin{pmatrix}1&-2&1\\2&-3&1\\3&-6&a\end{pmatrix} &= y_2\big(\tfrac{2}{3}\big)\begin{pmatrix}1&-2&1\\3&-6&a\\0&1&1-\frac{2}{3}a\end{pmatrix}\\
&= y_2\big(\tfrac{2}{3}\big)y_1\big(\tfrac{1}{3}\big)\begin{pmatrix}3&-6&a\\0&0&1-\frac{1}{3}a\\0&1&1-\frac{2}{3}a\end{pmatrix}\\
&= y_2\big(\tfrac{2}{3}\big)y_1\big(\tfrac{1}{3}\big)y_2(0)\begin{pmatrix}3&-6&a\\0&1&1-\frac{2}{3}a\\0&0&1-\frac{1}{3}a\end{pmatrix}\\
&= y_2\big(\tfrac{2}{3}\big)y_1\big(\tfrac{1}{3}\big)y_2(0)h(\tfrac{1}{3},1,1)\begin{pmatrix}1&-2&\frac{a}{3}\\0&1&1-\frac{2}{3}a\\0&0&1-\frac{1}{3}a\end{pmatrix}\\
&= y_2\big(\tfrac{2}{3}\big)y_1\big(\tfrac{1}{3}\big)y_2(0)h(\tfrac{1}{3},1,1)\begin{pmatrix}1&0&0\\0&1&1-\frac{2}{3}a\\0&0&1-\frac{1}{3}a\end{pmatrix}x_{12}(-2)x_{13}(\tfrac{a}{3})\\
&= y_2\big(\tfrac{2}{3}\big)y_1\big(\tfrac{1}{3}\big)y_2(0)h(\tfrac{1}{3},1,1)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1-\frac{1}{3}a\end{pmatrix}x_{23}(1-\tfrac{2}{3}a)x_{12}(-2)x_{13}(\tfrac{a}{3})
\end{aligned}$$

Thus, if

$$A=\begin{pmatrix}1&-2&1\\2&-3&1\\3&-6&a\end{pmatrix}\qquad\text{then}\qquad A=P1_rQ,\qquad\text{where}\quad r=\begin{cases}3,&\text{if } a\neq 3,\\2,&\text{if } a=3,\end{cases},$$