Topic 1. Example 10. Let us solve the equation

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

Multiply both sides by $y_1(1)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 6 \\ 1 \end{pmatrix}.$$

Multiply both sides by $y_3(0)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \\ 6 \end{pmatrix}.$$

Multiply both sides by $y_2(1)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \\ 6 \end{pmatrix}.$$

Multiply both sides by $y_3(1)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \\ 0 \end{pmatrix}.$$

Multiply both sides by $x_{13}(-1)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix}, \qquad \text{which gives} \qquad \begin{matrix} a = 9 - d, \\ b = 1 + d, \\ c = 6 - d, \\ 0 \cdot d = 0, \end{matrix}$$

with no restrictions on d. So

$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \ \Big|\ d \in \mathbb{F} \right\} = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \mathrm{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Normal form:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} = y_1(1) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$
$$= y_1(1)y_3(0) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$= y_1(1)y_3(0)y_2(1) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$= y_1(1)y_3(0)y_2(1)y_3(1) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= y_1(1)y_3(0)y_2(1)y_3(1)x_{13}(-1) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= y_1(1)y_3(0)y_2(1)y_3(1)x_{13}(-1)1_3x_{14}(1)x_{24}(-1)x_{34}(1) = P1_3Q,$$

where $P = y_1(1)y_3(0)y_2(1)y_3(1)x_{13}(-1)$,

$$1_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this case

$$P^{-1} = x_{13}(-1)^{-1}y_3(1)^{-1}y_2(1)^{-1}y_3(0)^{-1}y_1(1)^{-1} \qquad \text{and} \qquad Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$P^{-1} \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Sol}(Ax = b) = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \mathrm{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$