9 $\mathbb{F}$-modules

9.1 Vector spaces and linear transformations

Let $\mathbb{F}$ be a field. A $\mathbb{F}$-*vector space*, or $\mathbb{F}$-*module*, is a set V with functions

$$\begin{matrix} V\times V & \to & V \\ (v_1,v_2) & \mapsto & v_1+v_2 \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{F}\times V & \to & V \\ (c,v) & \mapsto & cv \end{matrix}$$

(*addition* and *scalar multiplication*) such that

(a) If $v_1, v_2, v_3 \in V$ then $(v_1+v_2)+v_3 = v_1+(v_2+v_3)$,
(b) There exists $0 \in V$ such that if $v \in V$ then $0+v=v$ and $v+0=v$,
(c) If $v\in V$ then there exists $-v \in V$ such that $v+(-v)=0$ and $(-v)+v=0$,
(d) If $v_1, v_2 \in V$ then $v_1+v_2=v_2+v_1$,
(e) If $c\in\mathbb{F}$ and $v_1,v_2\in V$ then $c(v_1+v_2)=cv_1+cv_2$,
(f) If $c_1,c_2\in\mathbb{F}$ and $v\in V$ then $(c_1+c_2)v=c_1v+c_2v$,
(g) If $c_1,c_2\in\mathbb{F}$ and $v\in V$ then $c_1(c_2v)=(c_1c_2)v$,
(h) If $v\in V$ then $1v=v$.

Linear transformations. Linear transformations are for comparing vector spaces.

Let $\mathbb{F}$ be a field and let V and W be $\mathbb{F}$-vector spaces. An $\mathbb{F}$-*linear transformation from V to W* is a function $f\colon V\to W$ such that

(a) If $v_1,v_2\in V$ then $f(v_1+v_2)=f(v_1)+f(v_2)$,
(b) If $c\in\mathbb{F}$ and $v\in V$ then $f(cv)=cf(v)$.

Subspaces. One vector space can be a subspace of another.

Let V be an $\mathbb{F}$-vector space. A *subspace of* V is a subset $W\subseteq V$ such that

(a) If $w_1,w_2\in W$ then $w_1+w_2\in W$,
(b) $0\in W$,
(c) If $w\in W$ then $-w\in W$,
(d) If $w\in W$ and $c\in\mathbb{F}$ then $cw\in W$.

The zero subspace. The tiniest vector space is the zero space.

The *zero space*, (0), is the set containing only 0 with the operations $0+0=0$ and $c\cdot 0$, for $c\in\mathbb{F}$.

9.2 Kernels and images

The *kernel*, or *null space*, of an $\mathbb{F}$-linear transformation $f\colon V\to W$ is the set

$$\ker(f)=\{v\in V \mid f(v)=0\}.$$

The *image* of an $\mathbb{F}$-linear transformation $f\colon V\to W$ is the set

$$\mathrm{im}(f)=\{f(v)\mid v\in V\}.$$

Proposition 9.1. *Let $f\colon V\to W$ be an $\mathbb{F}$-linear transformation. Then*

(a) $\ker f$ *is a subspace of* V*, and*
(b) $\mathrm{im}\, f$ *is a subspace of* W*.*

Let S and T be sets and let $f\colon S \to T$ be a function.
The function $f\colon S \to T$ is *injective* if f satisfies:

$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then } s_1 = s_2.$$

The function $f\colon S \to T$ is *surjective* if f satisfies:

$$\text{if } t \in T \text{ then there exists } s \in S \text{ such that } f(s) = t.$$

Proposition 9.2. *Let* $f\colon V \to W$ *be a linear transformation. Then*

(a) $\ker f = \{0\}$ *if and only if* f *is injective, and*
(b) $\mathrm{im}\, f = W$ *if and only if* f *is surjective.*

9.3 Bases and dimension

Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Let $\{v_1, v_2, \ldots, v_k\}$ be a subset of V.

- The *span* of the set $\{v_1, \ldots, v_k\}$ is

$$\mathrm{span}\{v_1, \ldots, v_k\} = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k \mid c_1, c_2, \ldots, c_k \in \mathbb{F}\}.$$

- A *linear combination* of $v_1, v_2, \ldots, v_k$ is an element of $\mathrm{span}\{v_1, \ldots, v_k\}$.
- The set $\{v_1, \ldots, v_k\}$ is *linearly independent* if it satisfies:

$$\text{if } c_1, \ldots, c_k \in \mathbb{F} \text{ and } c_1v_1 + \cdots + c_kv_k = 0 \quad \text{then} \quad c_1 = 0,\ c_2 = 0,\ \ldots,\ c_k = 0.$$

- A *basis* of V is a subset $B \subseteq V$ such that
  (a) $\mathrm{span}(B) = V$,
  (b) B is linearly independent.
- The *dimension* of V is the cardinality (number of elements) of a basis of V.

Proposition 9.3. *Let* V *be a vector space and let* B *be a subset of* V*. The following are equivalent:*
(a) B *is a basis of* V*;*
(b) B *is a minimal element of* $\{S \subseteq V \mid \mathrm{span}(S) = V\}$*, ordered by inclusion;*
(c) B *is a maximal element of* $\{L \subseteq V \mid L \text{ is linearly independent}\}$*, ordered by inclusion.*

Theorem 9.4. *Let* V *be a vector space over a field* $\mathbb{F}$*. Then*
(a) V *has a basis, and*
(b) Any two bases of V *have the same number of elements.*

9.4 Addition, scalar multiplication and composition of linear transformations

The *sum* of two $\mathbb{F}$-linear transformations $f_1 \colon V \to W$ and $f_2 \colon V \to W$ is the $\mathbb{F}$-linear transformation $(f_1 + f_2) \colon V \to W$.

$$(f_1 + f_2)(v) = f_1(v) + f_2(v), \qquad \text{for } v \in V.$$

Let $f \colon V \to W$ be an $\mathbb{F}$-linear transformation and let $c \in \mathbb{F}$. The *scalar multiplication* of f by c is the $\mathbb{F}$-linear transformation $(cf) \colon V \to W$ given by

$$(cf)(v) = c \cdot f(v), \qquad \text{for } v \in V.$$

The *composition* of an $\mathbb{F}$-linear transformation $f_2 \colon V \to W$ and an $\mathbb{F}$-linear transformation $f_1 \colon W \to Z$ is the $\mathbb{F}$-linear transformation $(f_1 \circ f_2) \colon V \to Z$ given by

$$(f_1 \circ f_2)(v) = f_1(f_2(v)), \qquad \text{for } v \in V.$$

9.5 Matrices of linear transformations and change of basis matrices

Let V and W be $\mathbb{F}$-vector spaces. Let B be a basis of V and let C be a basis of W. Let $f \colon V \to W$ be an $\mathbb{F}$-linear transformation. The *matrix of $f \colon V \to W$ with respect to the bases B and C* is the matrix

$$f_{CB} \in M_{C\times B}(\mathbb{F}) \qquad \text{given by} \qquad f(b) = \sum_{c\in C} f_{CB}(c,b)c \quad \text{for } b \in B.$$

Proposition 9.5. *Let V and W and Z be $\mathbb{F}$-vector spaces with bases B, C and D, respectively. Let*

$$f \colon V \to W, \quad g \colon V \to W, \quad h \colon W \to Z \quad \textit{be } \mathbb{F}\textit{-linear transformations}$$

and let $c \in \mathbb{F}$. Then

$$(cf)_{CB} = c \cdot f_{CB}, \qquad f_{CB} + g_{CB} = (f+g)_{CB} \qquad \textit{and} \qquad (h \circ g)_{DB} = h_{DC} g_{CB}.$$

Let V be an $\mathbb{F}$-vector space and let B and C be bases of V. The *change of basis matrix from B to C* is the matrix $P_{CB} \in M_{C\times B}(\mathbb{F})$ given by

$$b = \sum_{c\in C} P_{CB}(c,b)c, \qquad \text{for } b \in B. \tag{9.1}$$

Proposition 9.6. *Let $g \colon V \to W$ and $f \colon V \to V$ be an $\mathbb{F}$-linear transformations. Let*

$$B_1 \textit{ and } B_2 \textit{ be bases of } V, \quad \textit{and let} \quad C_1 \textit{ and } C_2 \textit{ be bases of } W,$$

and let $P_{B_1B_2}$ and $P_{C_2C_1}$ be the change of basis matrices defined as in (9.1). Then

$$g_{C_2B_2} = P_{C_2C_1} g_{C_1B_1} P_{B_1B_2} \qquad \textit{and} \qquad f_{B_2B_2} = P^{-1}_{B_1B_2} f_{B_1B_1} P_{B_1B_2}.$$