4.3 Unipotent upper triangular matrices $U_n(\mathbb{F})$

Let $n \in \mathbb{Z}_{>0}$.

- An $n \times n$ *unipotent upper triangular matrix* is an $n \times n$ matrix A such that

$$\text{if } i, j \in \{1, \ldots, n\} \text{ then} \qquad A(i,j) = \begin{cases} 0, & \text{if } j > i, \\ 1, & \text{if } i = j. \end{cases}$$

- The *unipotent radical* is

$$U_n = \{n \times n \text{ unipotent upper triangular matrices}\}.$$

- The *root matrices in U_n* are the matrices

$$x_{ij}(c) = 1 + cE_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & c & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

for $c \in \mathbb{F}$ and $i, j \in \{1, \ldots, n\}$ with $i < j$.

Theorem 4.3. *The group $U_n(\mathbb{F})$ is presented by generators*

$$x_{ij}(c), \qquad \textit{for } c \in \mathbb{F} \textit{ and } i, j \in \{1, \ldots, n\} \textit{ with } i < j,$$

with relations

$$x_{ij}(c_1)x_{ij}(c_2) = x_{ij}(c_1 + c_2),$$

$$\begin{aligned} x_{ij}(c_1)x_{k\ell}(c_2) &= x_{k\ell}(c_2)x_{ij}(c_1), && \textit{if } i \neq j \textit{ and } j \neq k, \\ x_{ij}(c_1)x_{j\ell}(c_2) &= x_{j\ell}(c_2)x_{ij}(c_1)x_{i\ell}(c_1c_2), && \textit{if } i \neq \ell, \\ x_{ij}(c_1)x_{ki}(c_2) &= x_{ki}(c_2)x_{ij}(c_1)x_{kj}(-c_1c_2), && \textit{if } j \neq k. \end{aligned}$$