4.2 Permutation matrices S_n

Let $n \in \mathbb{Z}_{>0}$.

- An $n \times n$ *permutation matrix* is an $n \times n$ matrix such that
  - (a) There is exactly one nonzero entry in each row and each column,
  - (b) The nonzero entries are 1.
- The *symmetric group* is
$$S_n = \{n \times n \text{ permutation matrices}\}.$$
- The *transpositions in* S_n are the matrices $s_{ij} = 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}$,

$$s_{ij} = \begin{array}{cc} & \begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& 0 &&&& 1 &&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&& 1 &&&& 0 &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix} \end{array}, \quad \text{for } i, j \in \{1, \ldots, n\} \text{ with } i \neq j.$$

- The *simple transpositions in* S_n are the matrices $s_i = s_{i,i+1}$,

$$s_i = \begin{array}{cc} & \begin{array}{ccccccc} & & & i & i+1 & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 0 & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \end{array}, \qquad \text{for } i \in \{1, \ldots, n-1\}. \tag{4.1}$$

Proposition 4.2. *The symmetric group S_n is presented by generators $s_1, s_2, \ldots, s_{n-1}$ and relations*

$$s_i^2 = 1 \qquad \textit{and} \qquad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1} \qquad \textit{and} \qquad s_k s_\ell = s_\ell s_k, \tag{4.2}$$

for $i, j, k, \ell \in \{1, \ldots, n-1\}$ with $j \neq n-1$ and $k \neq \ell \pm 1$.

A key step of the proof is to write a permutation matrix in normal form.