6.2 Ordered bases and ordered orthonormal bases

Let $\mathbb{F}$ be a field and let $\overline{}\colon \mathbb{F}\to\mathbb{F}$ be a function such that if $c, c_1, c_2 \in \mathbb{F}$ then

$$\overline{c_1+c_2} = \overline{c_1}+\overline{c_2}, \qquad \overline{c_1c_2} = \overline{c_1}\,\overline{c_2} \qquad \text{and} \qquad \overline{\overline{c}} = c.$$

The standard symmetric inner product on $\mathbb{F}^n$ is $\langle,\rangle\colon \mathbb{F}^n\times\mathbb{F}^n\to\mathbb{F}$ given by

$$\langle u, v\rangle = v^t u = u_1v_1+\cdots+u_nv_n, \qquad \text{for} \qquad \begin{array}{l} u=(u_1,\ldots,u_n)^t\in\mathbb{F}^n, \text{ and} \\ v=(v_1,\ldots,v_n)^t\in\mathbb{F}^n. \end{array} \qquad \text{(stdSinnprod)}$$

The standard Hermitian inner product on $\mathbb{F}^n$ is $\langle,\rangle\colon \mathbb{F}^n\times\mathbb{F}^n\to\mathbb{F}$ given by

$$\langle u, v\rangle = \overline{v}^t u = u_1\overline{v_1}+\cdots+u_n\overline{v_n}, \qquad \text{for} \qquad \begin{array}{l} u=(u_1,\ldots,u_n)^t\in\mathbb{F}^n, \text{ and} \\ v=(v_1,\ldots,v_n)^t\in\mathbb{F}^n. \end{array} \qquad \text{(stdHinnprod)}$$

Let

$$\begin{aligned} GL_n(\mathbb{F}) &= \{P\in M_n(\mathbb{F}) \mid P^{-1} \text{ exists in } M_n(\mathbb{F})\}, \\ O_n(\mathbb{F}) &= \{Q\in M_n(\mathbb{F}) \mid QQ^t = 1\}, \\ U_n(\mathbb{F}) &= \{U\in M_n(\mathbb{F}) \mid U\overline{U}^t = 1\}. \end{aligned}$$

An *ordered basis of* $\mathbb{F}^n$ is a sequence $(p_1,\ldots,p_n)$ of elements of $\mathbb{F}^n$ such that

(a) (span) $\mathbb{F}^n = \{c_1p_1+\cdots+c_np_n \mid c_1,\ldots,c_n\in\mathbb{F}\}$, and
(b) (linear independence) If $c_1,\ldots,c_n\in\mathbb{F}$ and

$$c_1p_1+\cdots+c_np_n = 0 \qquad \text{then} \qquad \text{if } k\in\{1,\ldots,n\} \text{ then } c_k=0.$$

An *ordered orthonormal basis of* $\mathbb{F}^n$ is an ordered basis $(u_1,\ldots,u_n)$ of $\mathbb{F}^n$ such that

$$\text{if } i,j\in\{1,\ldots,n\} \qquad \text{then} \qquad \langle u_i,u_j\rangle = \delta_{ij}.$$

Theorem 6.2.

(a) The function

$$\begin{array}{ccc} \left\{\begin{array}{l} \textit{ordered bases} \\ (p_1,\ldots,p_n) \textit{ of } \mathbb{F}^n \end{array}\right\} & \longrightarrow & GL_n(\mathbb{F}) \\ (p_1,\ldots,p_n) & \longmapsto & P = \begin{pmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{pmatrix} \end{array} \quad \textit{is a bijection.}$$

(b) Let $V=\mathbb{F}^n$ with inner product given by (stdSinnprod). The function

$$\begin{array}{ccc} \left\{\begin{array}{l} \textit{ordered orthonormal bases} \\ (q_1,\ldots,q_n) \textit{ of } \mathbb{F}^n \end{array}\right\} & \longrightarrow & O_n(\mathbb{F}) \\ (q_1,\ldots,q_n) & \longmapsto & Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \end{array} \quad \textit{is a bijection.}$$

(c) Let $V=\mathbb{F}^n$ with inner product given by (stdHinnprod). The function

$$\begin{array}{ccc} \left\{\begin{array}{l} \textit{ordered orthonormal bases} \\ (u_1,\ldots,u_n) \textit{ of } \mathbb{F}^n \end{array}\right\} & \longrightarrow & U_n(\mathbb{F}) \\ (u_1,\ldots,u_n) & \longmapsto & U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \end{array} \quad \textit{is a bijection.}$$