2.2.2 Normal form

The normal form algorithm (often called "row reduction"), when applied to a matrix $A \in M_{m\times n}(\mathbb{F})$ gives the following theorem.

Theorem 2.1. *Let $m, n \in \mathbb{Z}_{>0}$ and let $A \in M_{m\times n}(\mathbb{F})$. Then the normal form algorithm determines*

$$r \in \{0, \ldots, \min(m,n)\}, \quad \ell, s \in \mathbb{Z}_{>0}, \quad i_1, \ldots, i_\ell \in \{1, \ldots, n-1\}, \quad k_1, \ldots, k_s, \ell_1, \ldots, \ell_s \in \{1, \ldots, n\},$$

$$\gamma_1, \ldots \gamma_\ell, a_1, \ldots, a_s \in \mathbb{F}, \quad d_1, \ldots, d_r \in \mathbb{F}^\times, \quad J \subseteq \{1, \ldots, n\},$$

such that if

$$P = y_{i_1}(\gamma_1)\cdots y_{i_\ell}(\gamma_\ell)h(d_1, \ldots, d_r) \qquad \textit{and} \qquad Q = u_J x_{k_1\ell_1}(a_1)\cdots x_{k_s\ell_s}(a_s),$$

then

$$A = P1_rQ.$$

2.2.3 Orbit representatives for $GL_m(\mathbb{F})$ and $GL_n(\mathbb{F})$ acting on $M_{m\times n}(\mathbb{F})$

Let $m, n \in \mathbb{Z}_{>0}$. An $m \times n$ *reduced row echelon matrix* is a matrix $R \in M_{m\times n}(\mathbb{F})$ such that

(a) There exists $r \in \{1, \ldots, m\}$ such that rows $1, \ldots, r$ are not all zero and rows $r+1, \ldots, m$ are all 0,
(b) If $i \in \{1, \ldots, r-1\}$ then the first nonzero entry in each row i is 1,
(c) If $i \in \{1, \ldots, r-1\}$ and (i, c_i) is the position of the first nonzero entry in row i then all other entries in column c_i are 0,
(d) $c_1 < c_2 < \cdots < c_r$.

Theorem 2.2. *Let $\mathcal{E}$ be the set of reduced row echelon matrices in $M_{m\times n}(\mathbb{F})$ and*

$$\textit{let } 1_r \in M_{m\times n}(\mathbb{F}) \textit{ be given by} \qquad 1_r = E_{11} + \cdots + E_{rr},$$

where E_{ij} has 1 in the (i,j) entry and 0 elsewhere. Then

$$M_{m\times n}(\mathbb{F}) = \bigsqcup_{R\in\mathcal{E}} GL_m(\mathbb{F})\cdot R \qquad \textit{and} \qquad M_{m\times n}(\mathbb{F}) = \bigsqcup_{r=0}^{\min(m,n)} GL_m(\mathbb{F})1_rGL_n(\mathbb{F}),$$

where

$$GL_m(\mathbb{F})R = \{PR \in M_{m\times n}(\mathbb{F}) \mid P \in GL_m(\mathbb{F})\} \qquad \textit{and}$$
$$GL_m(\mathbb{F})1_rGL_n(\mathbb{F}) = \{P1_rQ \in M_{m\times n}(\mathbb{F}) \mid P \in GL_m(\mathbb{F}), Q \in GL_n(\mathbb{F})\}.$$