2.1 Examples of the steps in the normal form algorithm

2.1.1 Example for Step 1: increasing the number of lower left 0 entries

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2(\tfrac{1}{8}) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) y_3(\tfrac{4}{5}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix}$$

2.1.2 Example for Step 2: making the first entry of each row equal to 1

$$\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix} = h(8, 1, -\tfrac{5}{8}, -\tfrac{71}{40}) \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= h_1(8) h_2(1) h_3(-\tfrac{5}{8}) h_4(-\tfrac{71}{40}) \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combining examples 1.4.1 and 1.4.2 gives

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) y_3(\tfrac{4}{5}) h(8, 1, -\tfrac{5}{8}, -\tfrac{71}{40}) \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.1.3 Step 3 example: making entries above the first nonzero in each row equal to 0

$$\begin{pmatrix}1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8}\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & \frac{49}{5}\\ 0 & 0 & 0 & 1\end{pmatrix} = x_{34}(\tfrac{49}{5})\begin{pmatrix}1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8}\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = x_{34}(\tfrac{49}{5})x_{24}(2)\begin{pmatrix}1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8}\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})\begin{pmatrix}1 & \frac{6}{8} & \frac{3}{8} & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)\begin{pmatrix}1 & \frac{6}{8} & \frac{3}{8} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})\begin{pmatrix}1 & \frac{6}{8} & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})x_{12}(\tfrac{6}{8})\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Combining examples 1.4.1 and 1.4.2 and 1.4.3 gives

$$\begin{pmatrix}7 & 6 & 2 & 4\\ 1 & 8 & 7 & 9\\ 8 & 6 & 3 & 5\\ 0 & 1 & 1 & 2\end{pmatrix} = y_2(\tfrac{1}{8})y_1(\tfrac{7}{8})y_3(\tfrac{29}{4})y_2(\tfrac{3}{4})y_3(\tfrac{4}{5})h(8,1,-\tfrac{5}{8},-\tfrac{71}{40})x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})x_{12}(\tfrac{6}{8}).$$

2.1.4 Example for Step 4: making the upper right entries all equal to 0

$$\begin{pmatrix}1 & 0 & 0 & 5 & 0\\ 0 & 0 & 1 & 0 & 7\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 7\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} x_{14}(5)$$

$$= \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} x_{35}(7)x_{14}(5)$$

2.1.5 Example for Step 5: rearranging columns to get 1_r

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} s_{23} = 1_2 s_{23}$$

Combining examples 1.4.4 and 1.4.5 gives

$$\begin{pmatrix}1 & 0 & 0 & 5 & 0\\ 0 & 0 & 1 & 0 & 7\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} = 1_2 s_{23} x_{35}(7)x_{14}(5).$$