1 Matrices

1.1 Matrices and operations

Let $\mathbb{F}$ be a field. Let $m, n \in \mathbb{Z}_{>0}$.

- An $m \times n$ matrix with entries in $\mathbb{F}$ is a table of elements of $\mathbb{F}$ with m rows and n columns. More precisely, an $m \times n$ matrix with entries in $\mathbb{F}$ is a function

$$A \colon \{1, \ldots, m\} \times \{1, \ldots, n\} \longrightarrow \mathbb{F}.$$

- A *column vector of length* n is an $n \times 1$ matrix.
- A *row vector of length* n is an $1 \times n$ matrix.
- The (i, j) *entry of a matrix* A is the element $A(i, j)$ in row i and column j of A.

$$A = \begin{pmatrix} A(1,1) & A(1,2) & \cdots & A(1,m) \\ A(2,1) & A(2,2) & \cdots & A(2,m) \\ \vdots & & & \vdots \\ A(n,1) & A(n,2) & \cdots & A(n,m) \end{pmatrix}$$

Let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.
Let $M_n(\mathbb{F}) = M_{n\times n}(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$.

- The *sum* of $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$ given by

$$(A + B)(i, j) = A(i, j) + B(i, j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

- The *scalar multiplication* of an element $c \in \mathbb{F}$ with an $m \times n$ matrix A is the $m \times n$ matrix $c \cdot A$ given by

$$(c \cdot A)(i, j) = c \cdot A(i, j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

- The *product* of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix AB given by

$$\begin{aligned}(AB)(i, k) &= \sum_{j=1}^{n} A(i, j)B(j, k) \\ &= A(i, 1)B(1, k) + A(i, 2)B(2, k) + \cdots + A(i, n)B(n, k),\end{aligned}$$

  for $i \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, p\}$.

The *zero matrix* is the $m \times n$ matrix $0 \in M_{m\times n}(\mathbb{F})$ given by

$$0(i, j) = 0, \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

The *negative* of a matrix $A \in M_{m\times n}(\mathbb{F})$ is the matrix $-A \in M_{m\times n}(\mathbb{F})$ given by

$$(-A)(i, j) = -A(i, j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

For $k \in \{1, \ldots, m\}$ and $\ell \in \{1, \ldots, n\}$ let $E_{k\ell} \in M_{m\times n}(\mathbb{F})$ be the matrix given by

$$E_{k\ell}(i, j) = \begin{cases} 1, & \text{if } i = k \text{ and } j = \ell, \\ 0, & \text{otherwise,} \end{cases}$$

so that $E_{k\ell}$ has a 1 in the (k, ℓ) entry and all other entries 0.

Proposition 1.1. *Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.*

(a) If $A, B, C \in M_{m\times n}(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_{m\times n}(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_{m\times n}(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_{m\times n}(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A \in M_{m\times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$ then $c_1 \cdot (c_2 \cdot A) = (c_1c_2) \cdot A$.

(f) If $A \in M_{m\times n}(\mathbb{F})$ and $1 \in \mathbb{F}$ is the identity in $\mathbb{F}$ then $1 \cdot A = A$.

The *Kronecker delta* is given by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The *identity matrix* is the $n \times n$ matrix $1 \in M_{n\times n}(\mathbb{F})$ given by

$$1(i, j) = \delta_{ij}, \qquad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

Proposition 1.2. *Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B, C \in M_n(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_n(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_n(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_n(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A, B, C \in M_n(\mathbb{F})$ then $A(BC) = (AB)C$.

(f) If $A, B, C \in M_n(\mathbb{F})$ then $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.

(g) If $A \in M_n(\mathbb{F})$ then $1A = A$ and $A1 = A$.

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^t given by

$$A^t(i, j) = A(j, i), \qquad \text{for } i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, m\}.$$

Proposition 1.3. *Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$, and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B \in M_{m\times n}(\mathbb{F})$ then $(A + B)^t = A^t + B^t$,

(b) If $A \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$ then $(c \cdot A)^t = c \cdot A^t$,

(c) If $A, B \in M_n(\mathbb{F})$ then $(AB)^t = B^tA^t$.

(d) If $A \in M_n(\mathbb{F})$ then $(A^t)^t = A$.

Proposition 1.4. *Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then*

(a) (span) $M_{m\times n}(\mathbb{F}) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}E_{ij} \mid c_{ij} \in \mathbb{F} \right\}$.

(b) (linear independence) *If $c_{11}, \dots, c_{mn} \in \mathbb{F}$ and*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}E_{ij} = 0 \qquad \textit{then} \qquad \textit{if } k \in \{1, \dots, m\} \textit{ and } \ell \in \{1, \dots, n\} \textit{ then } c_{k\ell} = 0.$$