9.6 Some proofs

Proposition 9.7. *Let $T: V \to W$ be an $\mathbb{F}$-linear transformation. Let 0_V and 0_W be the zeros for V and W respectively. Then*

(a) $T(0_V) = 0_W$, *and*

(b) If $v \in V$ *then* $T(-v) = -T(v)$.

Proof.

(a) Add $-T(0_V)$ to both sides of the following equation,

$$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V).$$

(b) Since $T(v) + T(-v) = T\big(v + (-v)\big) = T(0_V) = 0_W$ and

$$T(-v) + T(v) = T\big((-v) + v\big) + T(0_V) = 0_W$$

then $-T(v) = T(-v)$.

□

Proposition 9.8. *Let $T: V \to W$ be an $\mathbb{F}$-linear transformation. Then*

(a) $\ker T$ *is a subspace of* V.

(b) $\mathrm{im}\ T$ *is a subspace of* W.

Proof. Let 0_V and 0_W be the zeros in V and W, respectively.

(a) By condition (a) in the definition of linear transformation, T is a group homomorphism.
To show: (aa) If $k_1, k_2 \in \ker T$ then $k_1 + k_2 \in \ker T$.
(ab) $0_V \in \ker T$.
(ac) If $k \in \ker T$ then $-k \in \ker T$.
(ad) If $c \in \mathbb{F}$ and $k \in \ker T$ then $ck \in \ker T$.

(aa) Assume $k_1, k_2 \in \ker T$.
Then $T(k_1) = 0_W$ and $T(k_2) = 0_W$.
By condition (a) in the definition of a linear transformation,

$$T(k_1 + k_2) = T(k_1) + T(k_2) = 0 + 0 = 0.$$

So $k_1 + k_2 \in \ker T$.

(ab) By Proposition 9.7(a), $T(0_V) = 0_W$.
So $0_V \in \ker T$.

(ac) Assume $k \in \ker T$.
By Proposition 9.7(b), $T(-k) = -T(k)$.
So $T(-k) = -T(k) = -0_W = 0_W$, and $-0_W = 0_W$ since $0_W + 0_W = 0_W$.
So $-k \in \ker T$.

(ad) Assume $c \in \mathbb{F}$ and $k \in \ker T$.

Then, by the definition of linear transformation,

$$T(ck) = cT(k) = c\,0_W = 0_W, \qquad \text{and} \quad c\,0_W = 0_W,$$

by adding $-c\,0_W$ to each side of $c\,0_W + c\,0_W = c(0_W + 0_W) = c\,0_W$.
So $T(ck) = 0_W$ and $ck \in \ker T$.

So $\ker T$ is a subspace of V.

(b) By condition (a) in the definition of an $\mathbb{F}$-linear transformation, T is a group homomorphism.
To show: (ba) If $w_1, w_2 \in \operatorname{im} T$ then $w_1 + w_2 \in \operatorname{im} T$.
(bb) $0_W \in \operatorname{im} T$.
(bc) If $w \in \operatorname{im} T$ then $-w \in \operatorname{im} T$.
(bd) If $c \in \mathbb{F}$ and $w \in \operatorname{im} T$ then $ck \in \operatorname{im} T$.

(ba) Assume $w_1, w_2 \in \operatorname{im} T$.
Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$.
By condition (a) in the definition of an $\mathbb{F}$-linear transformation,

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2.$$

So $w_1 + w_2 \in \operatorname{im} T$.
(bb) By Proposition 9.7(a), $T(0_V) = 0_W$.
So $0_W \in \operatorname{im} T$.
(bc) Assume $w \in \operatorname{im} T$.
The there exists $v \in V$ such that $T(v) = w$.
By Proposition 9.7(b), $T(-v) = -T(v) = -w$.
So $-w \in \operatorname{im}\ T$.
(bd) To show: If $c \in \mathbb{F}$ and $a \in \operatorname{im} T$ then $ca \in \operatorname{im} T$.
Assume $c \in \mathbb{F}$ and $c \in \operatorname{im} T$.
Then there exists $v \in V$ such that $a = T(v)$.
By the definition of an $\mathbb{F}$-linear transformation,

$$ca = cT(v) = T(cv).$$

So $ca \in \operatorname{im} T$.

So $\operatorname{im} T$ is a subspace of W.

□

Proposition 9.9. *Let $T\colon V \to W$ be an $\mathbb{F}$-linear transformation. Let 0_V be the zero in V. Then*

(a) $\ker T = (0_V)$ *if and only if T is injective.*
(b) $\operatorname{im} T = W$ *if and only if T is surjective.*

Proof. Let 0_V and 0_W be the zeros in V and W respectively.

(a) $\Longrightarrow$: Assume $\ker T = (0_V)$.
To show: If $T(v_1) = T(v_2)$ then $v_1 = v_2$.
Assume $T(v_1) = T(v_2)$.

Since T is an $\mathbb{F}$-linear transformation then

$$0_W = T(v_1) - T(v_2) = T(v_1 - v_2).$$

So $v_1 - v_2 \in \ker T$.
Since $\ker T = (0_V)$ then $v_1 - v_2 = 0_V$.
So $v_1 = v_2$.
So T is injective.

$\Longleftarrow$: Assume T is injective
To show: (aa) $(0_V) \subseteq \ker T$.
(ab) $\ker T \subseteq (0_V)$.

(aa) Since $T(0_V) = 0_W$ then $0_V \in \ker T$.
So $(0_V) \subseteq \ker T$.
(ab) Let $k \in \ker T$.
Then $T(k) = 0_W$.
So $T(k) = T(0_V)$.
Thus, since T is injective then $k = 0_V$.
So $\ker T \subseteq (0_V)$.

So $\ker T = (0_V)$.

(b) $\Longrightarrow$: Assume $\operatorname{im} T = W$.
To show: If $w \in W$ then there exists $v \in V$ such that $T(v) = w$.
Assume $w \in W$.
Then $w \in \operatorname{im} T$.
So there exists $v \in V$ such that $T(v) = w$.
So T is surjective.

$\Longleftarrow$: Assume T is surjective.
To show: (ba) $\operatorname{im} T \subseteq W$.
(bb) $W \subseteq \operatorname{im} T$.

(ba) Let $x \in \operatorname{im} T$.
Then there exists $v \in V$ such that $x = T(v)$.
By the definition of T, $T(v) \in W$.
So $x \in W$.
So $\operatorname{im} T \subseteq W$.

(bb) Assume $x \in W$.
Since T is surjective there exists $v \in V$ such that $T(v) = x$.
So $x \in \operatorname{im} T$.
So $W \subseteq \operatorname{im} T$.

So $\operatorname{im} T = W$.

$\square$

Proposition 9.10. *Let V be an $\mathbb{F}$-vector space and let B be a subset of V. The following are equivalent:*

(a) B is a basis of V.
(b) B is a minimal element of $\{S \subseteq V \mid span_{\mathbb{F}}(S) = V\}$.
(c) B is a maximal element of $\{L \subseteq V \mid L \text{ is linearly independent}\}$.

(In (b) and (c) the ordering is by inclusion.)

Proof.
(b) $\Rightarrow$ (a): Let $S \subseteq V$ such that $\mathrm{span}_{\mathbb{F}}(S) = V$.

To show: If S is minimal such that $\mathrm{span}_{\mathbb{F}}(V)$ then S is a basis.
To show: If S is minimal such that $\mathrm{span}_{\mathbb{F}}(V)$ then S is linearly independent.
Proof by contrapositive.
To show: If S is not linearly independent then S is not minimal such that $\mathrm{span}_{\mathbb{F}}(S) = V$.
Assume S is not linearly independent.
To show: There exists $s \in S$ such that $\mathrm{span}_{\mathbb{F}}(S - \{s\}) = V$.
Since S is linearly independent then there exist $k \in \mathbb{Z}_{>0}$ and $s_1, \ldots, s_k \in S$ and $c_1, \ldots, c_k \in \mathbb{F}$ and $i \in \{1, \ldots, k\}$ such that $c_1s_1 + \cdots + c_ks_k = 0$ and $c_i \neq 0$.
Let $s = s_i$.
Using that $\mathbb{F}$ is a field and $c_i \neq 0$ then

$$\begin{aligned} s = s_i &= c_i^{-1}(c_1s_1 + \ldots + c_{i-1}s_{i-1} + c_{i+1}s_{i+1} + \cdots + s_kc_k) \\ &= c_i^{-1}c_1s_1 + \cdots + c_i^{-1}c_{i-1}s_{i-1} + c_i^{-1}c_{i+1}s_{i+1} + \cdots + c_i^{-1}c_ks_k. \end{aligned}$$

So $V = \mathrm{span}_{\mathbb{F}}(S) = \mathrm{span}_{\mathbb{F}}(S - \{s\})$.
So S is not minimal such that $\mathrm{span}_{\mathbb{F}}(S) = V$.

(a) $\Rightarrow$ (b): Proof by contrapositive.

To show: If B is not minimal element of $\{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$ then B is not a basis of V.
Assume B is not minimal element of $\{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$.
So there exists $b \in B$ such that $\mathrm{span}_{\mathbb{F}}(B - \{b\}) \neq V$.
To show: (aa) $B \in \{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$.
(ab) If $b \in B$ then $B - \{b\} \notin \{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$.
(aa) Since $\mathrm{span}_{\mathbb{F}}(B) = V$ then $B \in \{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$.
(ab) Assume $b \in B$.
To show: $B - \{b\} \notin \{S \subseteq V \mid \mathrm{span}_{\mathbb{F}}(S) = V\}$.
To show: $\mathrm{span}_{\mathbb{F}}(B - \{b\}) \neq V$.
Since $\mathrm{span}_{\mathbb{F}}(B) = V$ then there exist $k \in \mathbb{Z}_{>0}$, $b_1, \ldots, b_k \in B$ and $c_1, \ldots, c_k \in \mathbb{F}$ such that $b = c_1b_1 + \cdots c_kb_k$.
So $0 = c_1b_1 + \cdots + c_kb_k + (-1)b$.

(a) $\Rightarrow$ (c): Assume B is a basis of V.

Since B is linearly independent then $B \in \{L \subseteq V \mid L \text{ is linearly independent}\}$.
To show: If $v \in V$ and $v \notin B$ then $B \cup \{v\}$ is not linearly independent.

Assume $v \in V$ and $v \notin B$.
Since $\mathrm{span}_{\mathbb{F}}(B) = V$ then there exists $k \in \mathbb{Z}_{>0}$ and $b_1, \ldots, b_k \in B$ and $c_1, \ldots, c_k \in \mathbb{F}$ such that $v = c_1 b_1 + \ldots + c_k b_k$.
So $0 = c_1 b_1 + \cdots + c_k b_k + (-1)v$.
So $B \cup \{v\}$ is not linearly independent.

(c) $\Rightarrow$ (a): Assume S is a maximal element of $\{L \subseteq V \mid L \text{ is linearly independent}\}$.

To show: $\mathrm{span}_{\mathbb{F}}(S) = V$.
To show: $V \subseteq \mathrm{span}_{\mathbb{F}}(S)$.
Let $v \in V$.
To show: $v \in \mathrm{span}_{\mathbb{F}}(S)$.
Case 1: $v \in S$. Then $v \in \mathrm{span}_{\mathbb{F}}(S)$.
Case 2: $v \notin S$.
Then $S \cup \{v\}$ is not linearly independent and S is linearly independent.
So there exist $k \in \mathbb{Z}_{>0}$ and $s_1, \ldots, s_k \in S$ and $c_0, c_1, \ldots, c_k \in \mathbb{F}$ such that

$$c_0 \neq 0 \qquad \text{and} \qquad c_0 v + c_1 s_1 + \cdots + c_k s_k = 0.$$

Since $\mathbb{F}$ is a field and $c_0 \neq 0$ then

$$v = (-c_0^{-1} c_1) s_1 + \cdots + (-c_0^{-1} c_k) s_k.$$

So $v \in \mathrm{span}_{\mathbb{F}}(S)$.
So $V \subseteq \mathrm{span}_{\mathbb{F}}(S)$ and $V = \mathrm{span}_{\mathbb{F}}(S)$.
So S is linearly independent and $\mathrm{span}_{\mathbb{F}}(S) = V$.
So S is a basis of V.

□

Theorem 9.11. *Let V be an $\mathbb{F}$-vector space. Then*

(a) V has a basis, and
(b) Any two bases of V have the same number of elements.

Proof.

(a) The idea is to use Zorn's lemma on the set $\{L \subseteq V \mid L \text{ is linearly independent}\}$, ordered by inclusion. We will not prove Zorn's lemma, we will assume it. Zorn's lemma is equivalent to the axiom of choice. For a proof see Isaacs book [Isa, §11D].

> **Zorn's Lemma.** *If S is a nonempty poset such that every chain in S has an upper bound then S has a maximal element.*

Let $v \in V$ such that $v \neq 0$.
Then $L = \{v\}$ is linearly independent.
So $\{L \subseteq V \mid L \text{ is linearly independent}\}$ is not empty.

To show: If $\cdots \subseteq S_{k-1} \subseteq S_k \subseteq S_{k+1} \subseteq \cdots$ chain of linearly independent subsets of V then there exists a linearly independent set S that contains all the S_k.
Assume $\cdots \subseteq S_{k-1} \subseteq S_k \subseteq S_{k+1} \subseteq \cdots$ is a chain of linearly independent subsets of V.
Let $L = \bigcup_k S_k$.
To show L is linearly independent.
Assume $\ell \in \mathbb{Z}_{>0}$ and $s_1, \ldots, s_\ell \in L$.
Then there exists k such that $s_1, \ldots, s_\ell \in S_k$.
Since S_k is linearly independent then if $c_1, \ldots, c_\ell \in \mathbb{F}$ and $c_1 s_1 + \cdots + c_\ell s_\ell = 0$ then $c_1 = 0$, $c_2 = 0, \ldots, c_\ell = 0$.
So L is linearly independent.
So, if $\cdots \subseteq S_{k-1} \subseteq S_k \subseteq S_{k+1} \subseteq \cdots$ chain of linearly independent subsets of V then there exists a linearly independent set B that contains all the S_k.
Thus, by Zorn's lemma, $\{L \subseteq V \mid L \text{ is linearly independent}\}$ has a maximal element B.
By Proposition 9.3, B is a basis of V.

(b) Let B and C be bases of V.

Case 1: V has a basis B with $\mathrm{Card}(B) < \infty$.
Let $b \in B$.
Then there exists $c \in C$ such that $c \notin \mathrm{span}_{\mathbb{F}}(B - \{b\})$.
Then $B_1 = (B - \{b\}) \cup \{c\}$ is a basis with the same cardinality as B.
Since B is finite then, by repeating this process, we can, after a finite number of steps, create a basis B' of V such that $B' \subseteq C$ and $\mathrm{Card}(B') = \mathrm{Card}(B)$.
Thus $\mathrm{Card}(B) = \mathrm{Card}(B') \leq \mathrm{Card}(C)$.
A similar argument with C in place of B gives that $\mathrm{Card}(B) \geq \mathrm{Card}(C)$.
So $\mathrm{Card}(B) = \mathrm{Card}(C)$.

Case 2: V has an infinite basis B.
Let C be a basis of V.
Define $P_{cb} \in \mathbb{F}$ for $c \in C$ and $b \in B$ by

$$b = \sum_{c \in C} P_{cb} c, \quad \text{and let} \qquad S_b = \{c \in C \mid P_{cb} \neq 0\} \quad \text{for } b \in B.$$

If $b \in B$ then S_b is a finite subset of C and

$$C = \bigcup_{b \in B} S_b, \qquad \text{since } C \text{ is a minimal spanning set.}$$

So $\mathrm{Card}(C) \leq \max\{\mathrm{Card}(S_b) \mid b \in B\} \leq \aleph_0 \mathrm{Card}(B)$.
A similar argument with B and C switched shows that $\mathrm{Card}(B) \leq \aleph_0 \mathrm{Card}(C)$.
So $\mathrm{Card}(C) \leq \aleph_0 \mathrm{Card}(B) = \mathrm{Card}(B) \leq \aleph_0 \mathrm{Card}(C) = \mathrm{Card}(C)$.
Since $\mathrm{Card}(C) \leq \mathrm{Card}(B) \leq \mathrm{Card}(C)$ then $\mathrm{Card}(C) = \mathrm{Card}(B)$.

$\square$

Proposition 9.12. *Let V and W and Z be $\mathbb{F}$-vector spaces with bases B, C and D, respectively. Let*

$$f\colon V \to W, \quad g\colon V \to W, \quad h\colon W \to Z \quad \text{be linear transformations}$$

and let $c \in \mathbb{F}$. Then

$$(cf)_{CB} = c \cdot f_{CB}, \qquad f_{CB} + g_{CB} = (f+g)_{CB} \qquad \text{and} \qquad (h \circ g)_{DB} = h_{DC} g_{CB}.$$

Proof. Let $b \in B$ and $c' \in C$. Taking the coefficient of c' on each side of

$$\sum_{c\in C}(\alpha f)_{CB}(c,b)c = (\alpha f)(b) = \alpha\cdot f(b) = \alpha\cdot\Big(\sum_{c\in C} f_{CB}(c,b)c\Big) = \sum_{c\in C}\alpha f_{CB}(c,b)c$$

gives $(\alpha f)_{CB}(c',b) = \alpha\cdot f_{CB}(c',b)$.
So $(\alpha f)_{CB} = \alpha\cdot f_{CB}$.
Let $b \in B$ and $c' \in C$. Taking the coefficient of c' on each side of

$$\begin{aligned}\sum_{c\in C}(f+g)_{CB}(c,b)c = (f+g)(b) = f(b)+g(b) &= \sum_{c\in C}(f_{CB}(c,b)c + \sum_{c\in C} g_{CB}(c,b)c \\ &= \sum_{c\in C}(f_{CB}(c,b)c + g_{CB}(c,b)c = \sum_{c\in C}(f_{CB}(c,b)+g_{CB}(c,b))c\end{aligned}$$

gives $(f_{CB}+g_{CB})(c',b) = f_{CB}(c',b)+g_{CB}(c',b)$.
So $f_{CB}+g_{CB} = (f+g)_{CB}$.
Let $b \in B$ and $d' \in D$. Taking the coefficient of d' on each side of

$$\begin{aligned}\sum_{d\in D}(h\circ g)_{DB}(d,b)d = (h\circ g)(b) = h(g(b)) &= h\Big(\sum_{c\in C} g_{CB}(c,b)c\Big) \\ &= \sum_{c\in C} g_{CB}(c,b)h(c) = \sum_{c\in C}\sum_{d\in D} g_{CB}(c,b)h_{DC}(d,c)d,\end{aligned}$$

gives $(h\circ g)_{DB}(d',b) = \sum_{c\in C}\sum_{d\in D} h_{DC}(d,'c)g_{CB}(c,b) = (h_{DC}g_{CB})(d',b)$.
So $(h\circ g)_{DB} = (h_{DC}g_{CB})$. □

Proposition 9.13. *Let $g\colon V\to W$ and $f\colon V\to V$ be $\mathbb{F}$-linear transformations. Let*

$$B_1 \text{ and } B_2 \text{ be bases of } V, \quad \text{and let} \quad C_1 \text{ and } C_2 \text{ be bases of } W,$$

and let $P_{B_1B_2}$ and $P_{C_2C_1}$ be the change of basis matrices defined as in (9.1). Then

$$g_{C_2B_2} = P_{C_2C_1}g_{C_1B_1}P_{B_1B_2} \qquad \text{and} \qquad f_{B_2B_2} = P_{B_1B_2}^{-1}f_{B_1B_1}P_{B_1B_2}.$$

Proof. Let $\beta,\beta' \in B_2$. Comparing coefficients of β' on each side of

$$\begin{aligned}\beta = \sum_{b\in B_1} P_{B_1B_2}(b,\beta)b &= \sum_{b\in B_1} P_{B_1B_2}(b,\beta)\sum_{\beta'\in B_2} P_{B_2B_1}(\beta',b)\beta' \\ &= \sum_{b\in B_1}\sum_{\beta'\in B_2} P_{B_2B_1}(\beta',b)P_{B_1B_2}(b,\beta)\beta' = \sum_{b\in B_1}\sum_{\beta'\in B_2}(P_{B_2B_1}P_{B_1B_2})(\beta',\beta)\beta'\end{aligned}$$

gives

$$(P_{B_2B_1}P_{B_1B_2})(\beta',\beta) = \delta_{\beta'\beta}.$$

So $P_{B_2B_1} = P_{B_1B_2}^{-1}$.
Let $\beta \in B_1$ and $c \in B_2$. Taking the coefficient of b' on each side of

$$f(c) = \sum_{c'\in B_2} f_{B_2B_2}(c',c)c' = \sum_{b'\in B_1} f_{B_2B_2}(c',c)P_{B_1B_2}(b',c')b'$$

and

$$f(c) = f\Big(\sum_{b\in B_1} P_{B_1B_2}(b,c)b\Big) = \sum_{b\in B_1} P_{B_1B_2}(b,c)f(b) = \sum_{b\in B_1} P_{B_1B_2}(b,c)\sum_{b'\in B_1} f_{B_1B_1}(b',b)b'$$

gives

$$(P_{B_1B_2}f_{B_2B_2})(\beta,b) = (f_{B_1B_1}P_{B_1B_2})(\beta,b).$$

So

$$P_{B_1B_2}f_{B_2B_2} = f_{B_1B_1}P_{B_1B_2} \qquad \text{and thus} \qquad f_{B_2B_2} = P_{B_1B_2}^{-1}f_{B_1B_1}P_{B_1B_2}.$$

Let $\gamma' \in C_2$ and $\beta \in B_2$. Taking the coefficient of γ on each side of

$$\begin{aligned}
\sum_{\gamma\in C_2} g_{C_2B_2}(\gamma,\beta)\gamma = g(\beta) = g(\sum_{b\in B_1} P_{B_1B_2}(b,\beta)b) &= \sum_{b\in B_1} P_{B_1B_2}(b,\beta)g(b) \\
&= \sum_{b\in B_1} P_{B_1B_2}(b,\beta)\sum_{c\in C_1} g_{C_1B_1}(c,b)c \\
&= \sum_{b\in B_1} P_{B_1B_2}(b,\beta)\sum_{c\in C_1} g_{C_1B_1}(c,b)\sum_{\gamma\in C_2} P_{C_2C_1}(\gamma,c)\gamma \\
&= \sum_{b\in B_1, c\in C_1, \gamma\in C_2} P_{C_2C_1}(\gamma,c)g_{C_1B_1}(c,b)P_{B_1B_2}(b,\beta)\gamma \\
&= \sum_{\gamma\in C_2} (P_{C_2C_1}g_{C_1B_1}P_{B_1B_2})(\gamma,\beta)\gamma
\end{aligned}$$

gives $g_{C_2B_2}(\gamma',\beta) = (P_{C_2C_1}g_{C_1B_1}P_{B_1B_2})(\gamma',\beta)$. So $g_{C_2B_2} = P_{C_2C_2}g_{C_1B_1}P_{B_1B_2}$. □

Proposition 9.14. *Let $P \in M_n(\mathbb{F})$. The matrix P is invertible if and only if the columns of P are linearly independent in $\mathbb{F}^n$.*

Proof.
$\Rightarrow$: Assume P is invertible. Let $p_1, \ldots, p_n$ be the columns of P.
To show: $\{p_1, \ldots, p_n\}$ is linearly independent.
Assume $c_1, \ldots, c_n \in \mathbb{F}$ and $c_1p_1 + \cdots + c_np_n = 0$.
Let $c = (c_1, \ldots, c_n)^t \in \mathbb{F}^n$.
Since $c_1p_1 + \cdots + c_np_n = 0$ then $Pc = 0$.
So $c = P^{-1}Pc = P^{-1}0 = 0$.
So $c_1 = 0, \ldots, c_n = 0$.

$\Leftarrow$: Assume the columns of P are linearly independent.
To show: There exists $Q \in M_n(\mathbb{F})$ such that $QP = 1$.
Let $p_1, \ldots, p_n$ be the columns of P.
Since $B = \{p_1, \ldots, p_n\}$ is linearly independent and $\dim(\mathbb{F}^n) = n$ then B is a maximal linearly independent set.
Thus, by Theorem 9.3, B is a basis.
Let $S = \{e_1, \ldots, e_n\}$ where e_i has 1 in the ith spot and 0 elsewhere.
Then $P = P_{BS}$, the change of basis matrix from S to B.
Let $Q = P_{SB}$, the change of basis matrix from B to S.
Then $QP = P_{SB}P_{BS} = P_{SS} = 1$.
So P is invertible. □