6 Eigenvalues, eigenvectors and diagonalization

6.1 Eigenvalues and eigenvectors

Let $A \in M_n(\mathbb{F})$.

- An *eigenvalue of* A is an element $\lambda \in \mathbb{F}$ such that $\ker(A - \lambda) \neq 0$.

Let $A \in M_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$.

- An *eigenvector of* A *of eigenvalue* λ is a nonzero element of $\ker(A - \lambda)$.

Theorem 6.1. *Let* $A \in M_n(\mathbb{F})$ *and let* $\lambda \in \mathbb{F}$. *Then*

$$\ker(A - \lambda) \neq 0 \qquad \textit{if and only if} \qquad \det(A - \lambda) = 0.$$

Proof. Use normal form to write $A - \lambda = P 1_r Q$ with $P, Q \in GL_n(\mathbb{F})$. Then use Proposition 3.4.
$\Rightarrow$: Assume $\ker(A - \lambda) \neq 0$.
Since $\ker(A - \lambda) \neq 0$ then $r < n$.
Since $\det(1_r) = 0$ then $\det(A - \lambda) = \det(P 1_r Q) = \det(P)\det(1_r)\det(Q) = 0$.
$\Leftarrow$: Assume that $\det(A - \lambda) = 0$.
Since P and Q are invertible matrices then $\det(P)$ and $\det(Q)$ are invertible elements of $\mathbb{F}$.
Since $\det(1_r) = \det(P^{-1}(A - \lambda)Q^{-1}) = \det(P)^{-1}\det(A - \lambda)\det(Q)^{-1} = \det(P)^{-1} \cdot 0 \cdot \det(Q)^{-1} = 0$ then $r < n$.
So $\ker(1_r) \neq 0$.
So $\ker(A - \lambda) = \ker(P 1_r Q) = Q^{-1}\ker(1_r) \neq 0$. □