5.5 Some proofs

5.5.1 Determinants of permutations

Proposition 5.9. *There are exactly two functions $f\colon S_n \to GL_1(\mathbb{F})$ which satisfy*

$$\text{if } w_1, w_2 \in S_n \quad \text{then} \qquad f(w_1w_2) = f(w_1)f(w_2),$$

the function triv$\colon S_n \to GL_1(\mathbb{F})$ *and the function* det$\colon S_n \to GL_1(\mathbb{F})$ *determined by*

$$\mathrm{triv}(s_i) = 1 \qquad \text{and} \qquad \det(s_i) = -1,$$

for $i \in \{1, \ldots, n\}$.

Proof. Assume $f\colon S_n \to GL_1(\mathbb{F})$ satisfies, if $w_1, w_2 \in S_n$ then $f(w_1w_2) = f(w_1)f(w_2)$. By Proposition 4.2, f is determined by the values $f(s_1), \ldots, f(s_{n-1})$ and these values satisfy

$$f(s_i)f(s_{i+1})f(s_i) = f(s_{i+1})f(s_i)f(s_{i+1}) \qquad \text{and} \quad f(s_i)^2 = 1,$$

for $i \in \{1, \ldots, n-1\}$. The equation $f(s_i)^2 = 1$ forces $f(s_1) = \pm 1$ and the equation $f(s_i)f(s_{i+1})f(s_i) = f(s_{i+1})f(s_i)f(s_{i+1})$ forces $f(s_i) = f(s_{i+1})$, So $f = \mathrm{triv}$ or $f = \det$. ☐

5.5.2 Determinants of square matrices

Theorem 5.10. *The functions $f\colon M_n(\mathbb{F}) \to \mathbb{F}$ which satsify*

$$\text{if } A, B \in M_n(\mathbb{F}) \quad \text{then} \quad f(AB) = f(A)f(B),$$

are the functions

$$\begin{array}{rccc} \det^k\colon & M_n(\mathbb{F}) & \longrightarrow & \mathbb{F} \\ & A & \longmapsto & \det(A)^k \end{array} \qquad \text{for } k \in \mathbb{Z},$$

where the function det$\colon M_n(\mathbb{F}) \to \mathbb{F}$ *is determined by*

$$\det(AB) = \det(A)\det(B)$$

and the conditions

$$\det(x_{ij}(c)) = 1, \quad \det(s_i) = -1, \quad \text{and} \qquad \det(h_i(d)) = d,$$

for $i, j \in \{1, \ldots, n\}$ with $i < j$, $c \in \mathbb{F}$ and $d \in \mathbb{F}$.

Proof. Since $1_r = h_{r+1}(0) \cdots h_n(0)$, then $\det(1_r) = 0$. Since the row reduction proces writes A as a product of elementary matrices and 1_r, the homomorphism property determines the value of $\det(A)$. The homomorphism is well-defined since the values for the determinants of the elementary matrices satisfy the relations in Theorem 4.4. ☐

5.5.3 Determinants of permutations

Let $n \in \mathbb{Z}_{>0}$. The *symmetric group* is

$$S_n = \{w\colon \{1, \ldots, n\} \to \{1, \ldots, n\} \mid w \text{ is a bijection}\}.$$

Identify w with the $n \times n$ matrix with

$$w(i,j) = \begin{cases} 1, & \text{if } j = w(i), \\ 0, & \text{otherwise.} \end{cases}$$

For $w \in W$ let

$$\mathrm{Inv}(w) = \{(i,j) \mid i,j \in \mathbb{Z}_{[1,k]},\ i<j,\ w(i) > w(j)\} \qquad \text{and} \qquad \ell(w) = \#\mathrm{Inv}(w).$$

Then

$$\det(w) = (-1)^{\ell(w)}.$$

THE PROOF FOLLOWS FROM

$$\text{If } \ell(ws_k) > \ell(w) \text{ then} \qquad \mathrm{Inv}(ws_k) = s_k\mathrm{Inv}(w) \cup \{(k,k+1)\}.$$

5.5.4 Determinants of square matrices: the permutation formula

For $A \in M_n(\mathbb{F})$ define

$$D(A) = \sum_{w\in S_n} \det(w) A(1,w(1))A(2,w(2))\cdots A(n,w(n)),$$

Proposition 5.11. *Let $A \in M_n(\mathbb{F})$ and let $i,j \in \{1,\ldots,n\}$ with $i<j$.*
(a) If u is a permutation then $D(uA) = \det(u)D(A)$.
(b) If row i and row j of A are equal then $D(A) = 0$.
(c) $D(AB) = D(A)D(B)$.
(d) $D(s_{ij}) = -1$, $D(x_{ij}(c)) = 1$ and $D(h_i(d)) = c$.

Proof. Let $u\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ be a bijection. Then

$$\begin{aligned} D(uA) &= \sum_{w\in S_n} \det(w)A(u(1),w(1))A(u(2),w(2))\cdots A(u(n),w(n)) \\ &= \sum_{w\in S_n} \det(wu)A(u(1),wu(1))A(u(2),wu(2))\cdots A(u(n),wu(n)) \\ &= \sum_{w\in S_n} \det(wu)A(1,w(1))A(2,w(2))\cdots A(n,w(n)) \\ &= \sum_{w\in S_n} \det(w)\det(u)A(1,w(1))A(2,w(2))\cdots A(n,w(n)) \\ &= \det(u)\sum_{w\in S_n} \det(w)A(1,w(1))A(2,w(2))\cdots A(n,w(n)) = \det(u)D(A). \end{aligned}$$

If row i and row j of A are equal then $A = s_{ij}A$. Using part (a),

$$D(A) = D(s_{ij}A) = \det(s_{ij})D(A) = -D(A), \qquad \text{giving} \qquad D(A) = 0. \qquad \text{(Deqrows)}$$

$$\begin{aligned}
D(AB) &= \sum_{w\in S_n} \det(w)(AB)(1,w(1))\cdots(AB)(n,w(n)) \\
&= \sum_{w\in S_n} \det(w) \sum_{k_1,\ldots,k_n=1}^{n} A(1,k_1)B(k_1,w(1))\cdots A(n,k_n)B(k_n,w(n)) \\
&= \sum_{k_1,\ldots,k_n=1}^{n} A(1,k_1)\cdots A(n,k_n) \sum_{w\in S_n} \det(w)B(k_1,w(1))\cdots B(k_n,w(n)).
\end{aligned}$$

By part (b), the second sum is zero if any two entries in $k = (k_1,\ldots,k_n)$ are the same. If all the entries in $k = (k_1,\ldots,k_n)$ are distinct and, by part (a), the second sum is $\det(k)D(B)$. Hence,

$$\begin{aligned}
D(AB) &= \sum_{k=(k_1,\ldots,k_n)\in S_n} \det(k)A(1,k_1)\cdots A(n,k_n)\det(k)D(B) \\
&= \sum_{k\in S_n} \det(k)A(1,k(1))\cdots A(n,k(n))D(B) = D(A)D(B).
\end{aligned}$$

For each of the expressions (d), the permutation expansion has exactly one nonzero term and this term equals the right hand side. □

Theorem 5.12. *Let $A \in M_n(\mathbb{F})$. Then*

$$\det(A) = \sum_{w\in S_n} \det(w)A(1,w(1))A(2,w(2)\cdots A(n,w(n)).$$

Proof. By Proposition 5.11, $D(A)$ satisfies the defining properties of $\det(A)$ which are given in Theorem 5.10. So $\det(A) = D(A)$. □

5.5.5 Laplace expansion

Let $J \subseteq \{1,\ldots,n\}$ with $|J| = k$. Write

$$\begin{matrix} J = \{j_1,\ldots,j_k\} \\ J^c = \{\ell_1,\ldots,\ell_{n-k}\} \end{matrix} \qquad \text{where} \qquad \begin{matrix} j_1 < \cdots < j_k \ \text{ and} \\ \ell_1 < \cdots < \ell_{n-k}, \end{matrix}$$

and define a permutation u_J by

$$u_J(r) = \begin{cases} j_r, & \text{if } r \in \{1,\ldots,k\}, \\ \ell_{r-k}, & \text{if } s \in \{k+1,\ldots,n\}. \end{cases}$$

Theorem 5.13. *Let $A \in M_n(\mathbb{F})$.*

(a) (General Laplace expansion) Let $K, L \subseteq \{1,\ldots,n\}$ with $|K| = |L| = k$. Then

$$\sum_{\substack{J\subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \det(u_J)\det(A_{K,J})\det(A^{(L,J)}) = \begin{cases} \det(u_K)\det(A), & \text{if } K = L, \\ 0, & \text{if } K \neq L. \end{cases}$$

where W^J is a set of coset representatives of cosets of S_n/W_J, $A_{K,J}$ is the submatrix of A consisting of entries of A in rows indexed by the elements of K and the entries in columns indexed by J and $A^{(L,J)}$ is the matrix obtained from A by removing the rows indexed by L and removing the columns indexed by elements of J.

(b) *(Laplace expansion on the kth row). Let $k, \ell \in \{1, \ldots, n\}$.*

$$\sum_{j=1}^{n} (-1)^{k+j} A(k,j) \det(A^{(j;\ell)}) = \begin{cases} \det(A), & \text{if } k = \ell, \\ 0, & \text{if } k \neq \ell. \end{cases}$$

Proof. The set $\{u_J \mid J \subseteq \{1, \ldots, n\}, |J| = k\}$ is the set of minimal length coset representatives of the cosets in $S_n/(S_k \times S_{n-k})$. Then

$$\begin{aligned}
\det(A) &= \sum_{w \in S_n} \det(w) A(1, w(1)) \cdots A(n, w(n)) \\
&= \sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \sum_{v \in S_k \times S_{n-k}} \det(u_J v) A(1, u_J v(1)) \cdots A(n, u_J v u(n)) \\
&= \sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \det(u_J) \sum_{v \in S_k \times S_{n-k}} \det(v) A(1, u_J v(1)) \cdots A(n, u_J v(n)) \\
&= \sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \det(u_J) \sum_{v \in S_k} \sum_{z \in S_{n-k}} \det(v) \det(z) \left(\begin{array}{c} \det(A(1, u_J v(1)) \cdots A(k, u_J v(k)) \\ \cdot A(k+1, k+z(1)) \cdots A(n, k+z(n-k)) \end{array} \right) \\
&= \sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \det(u_J) \det(A_{1..k,J}) \det(A_{k+1..n,J^c}). \qquad (5.1)
\end{aligned}$$

Now let K and L be subsets of $\{1, \ldots, n\}$ with $|K| = |L|$. Let $A_{KL^c,1..n}$ be the matrix formed by taking the rows of A labeled by the indices in K followed by the rows of A labeled by the indices in L^c. If $K = L$ then $A_{KL^c,1..n}$ is a rearrangement of the rows of A and if $K \neq L$ then $A_{KL^c,1..n}$ has two equal rows. Applying the identity (5.1) to the matrix A_{KL^c} gives

$$\sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J|=k}} \det(u_J) \det(A_{K,J}) \det(A_{L^c,J^c}) = \det(A_{KL^c,1..n}) = \begin{cases} 0, & \text{if } K \neq L, \\ \det(u_K) \det(A), & \text{if } K = L. \end{cases}$$

(b) is the special case of (a) where $|K| = |L| = 1$. If $K = \{k\}$ then $\det(u_K) = (-1)^{k-1}$ so that $\det(u_K) \det(u_J) = (-1)^{(k-1)+(j-1)} = (-1)^{k+j}$. □

5.5.6 Inverse by determinants

Theorem 5.14. *(Inverse by determinants) Let $A \in M_n(\mathbb{F})$ such that $\det(A)$ is invertible. Then the inverse of A is the matrix A^{-1} given by*

$$A^{-1}(i,j) = \frac{1}{\det(A)} (-1)^{i+j} \det(A^{(i;j)}).$$

where $A^{(i;j)}$ is the matrix A with the ith and the jth column removed.

Proof. Let $B \in M_n(\mathbb{F})$ be given by

$$B(i,j) = \frac{1}{\det(A)} (-1)^{i+j} \det(A^{(i;j)}).$$

To show: $AB = 1$ and $BA = 1$.
To show: If $i, j \in \{1, \ldots, n\}$ then $(AB)(i,j) = \delta_{ij}$ and $(BA)(i,j) = \delta_{ij}$.

Assume $i, j \in \{1, \ldots, n\}$.
To show: $(AB)(i,j) = \delta_{ij}$ and $(BA)(i,j) = \delta_{ij}$.

$$\begin{aligned}(AB)(i,j) &= \sum_{k=1}^{n} A(i,k)B(k,j) = \sum_{k=1}^{n} A(i,k)\frac{1}{\det(A)}(-1)^{j+k}\det(A^{(j;k)}) \\ &= \frac{1}{\det(A)}\sum_{k=1}^{n} A(i,k)(-1)^{j+k}\det(A^{(j;k)}) = \frac{1}{\det(A)}\det(A)\delta_{ij} = \delta_{ij},\end{aligned}$$

where the next to last equality is by Laplace expansion on the ith rowCHECK THIS, □

5.5.7 Cramer's rule

Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_n(\mathbb{F})$ and assume that A is invertible.

$$\text{Let} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \text{be elements of } \mathbb{F}^n \text{ with} \quad Ax = b.$$

For $i \in \{1, \ldots, n\}$ let

$$b \xrightarrow{i} A \qquad \text{be the matrix } A \text{ except with } i\text{th column replaced by } b.$$

Then

$$x_1 = \frac{\det(b \xrightarrow{1} A)}{\det(A)}, \quad x_2 = \frac{\det(b \xrightarrow{2} A)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(b \xrightarrow{n} A)}{\det(A)}.$$

Proof. Let $i \in \{1, \ldots, n\}$. Then

$$x_i = (A^{-1}b)_i = \frac{1}{\det(A)}\sum_{j=1}^{n}(-1)^{i+j}\det(A^{(i;j)})b_j = \frac{1}{\det(A)}\det(b \xrightarrow{i} A),$$

where the second equality is by (inverse by cofactors) and the third is by the Laplace expansion on the ith row. □

5.5.8 The Cayley-Hamilton theorem

Theorem 5.15. *(Cayley-Hamilton)*

(a)

$$\begin{aligned}\det(x - A) &= \sum_{k=0}^{n}(-1)^{n-k}\mathrm{tr}(A^{\wedge(n-k)})x^k \\ &= x^n - \mathrm{tr}(A^{\wedge 1})x^{n-1} + \mathrm{tr}(A^{\wedge 2})x^{n-2} - \cdots + (-1)^n\mathrm{tr}(A^{\wedge n}).\end{aligned}$$

(b) $\mathrm{ev}_A(\det(x - A)) = 0$.

Proof. (b) The Bourbaki reference for the Cayley-Hamilton theorem is [**?**, Ch. III no. 8 Prop. 20]. The proof from A.O. Morris: Write

$$\begin{aligned}&\det(A - x) = (-1)^n(x^n + a_1x^{n-1} + \cdots + a_n) \qquad \text{and} \\ &\mathrm{adj}(A - x) = (A - x)(B_0 + B_1x + \cdots + B_{n-1}x^{n-1}),\end{aligned}$$

where $a_0, \ldots, a_n \in \mathbb{F}$ and $B_0, \ldots, B_{n-1} \in M_n(\mathbb{F})$.

As elements of $M_n(\mathbb{F})[x]$:

$$\begin{aligned}(-1)^n a_n \cdot 1_n + (-1)^n a_{n-1} \cdot 1x + \cdots + (-1)^n a_1 \cdot 1_n x^{n-1} + (-1)^n \cdot 1x^n &= \det(A-x) \cdot 1_n \\ &= (A-x)\mathrm{adj}(A-x) = (A-x)(B_0 + B_1 x + \cdots + B_{n-1}x^{n-1}) \\ &= AB_0 + (-B_0 + AB_1)x \cdots (-B_{n-2} + AB_{n-1})x^{n-1} + (-B_{n-1})x^n.\end{aligned}$$

As elements of $M_n(\mathbb{F})$:

$$\begin{aligned} AB_0 &= (-1)^n a_n \cdot 1_n, \\ -B_0 + AB_1 &= (-1)^n a_{n-1} \cdot 1_n, \\ &\vdots \\ -B_{n-2} + AB_{n-1} &= (-1)^n a_1 \cdot 1_n, \\ -B_{n-1} &= (-1)^n \cdot 1_n. \end{aligned}$$

Thus

$$\begin{aligned}(-1)^n a_n \cdot 1_n &+ A(-1)^n a_{n-1} \cdot 1_n + \cdots + A^{n-1}(-1)^n a_1 \cdot 1_n + A^n(-1)^n \cdot 1_n \\ &= AB_0 + A(-B_0 + AB_1) + \cdots + A^{n-1}(-B_{n-2} + AB_{n-1}) + A^n(-B_{n-1}) \\ &= (A-A)(B_0 + B_1 A + \cdots + B_{n-1}A^{n-1}) \\ &= 0 \cdot (B_0 + B_1 A + \cdots + B_{n-1}A^{n-1}) = 0.\end{aligned}$$

So $\det(A-x) \in \ker(\varphi_A)$. $\square$