5 Determinants

5.1 Determinants are homomorphisms

5.1.1 Determinants of permutations

Let

$$GL_1(\mathbb{F}) = \mathbb{F}^\times = \{c \in \mathbb{F} \mid c \text{ is invertible}\} = \{c \in \mathbb{F} \mid c \neq 0\}.$$

Proposition 5.1. *There are exactly two functions* $f\colon S_n \to GL_1(\mathbb{F})$ *which satisfy*

$$\text{if } w_1, w_2 \in S_n \quad \text{then} \qquad f(w_1w_2) = f(w_1)f(w_2),$$

the function $\mathrm{triv}\colon S_n \to GL_1(\mathbb{F})$ *and the function* $\det\colon S_n \to GL_1(\mathbb{F})$ *determined by*

$$\mathrm{triv}(s_i) = 1 \qquad \text{and} \qquad \det(s_i) = -1, \qquad \text{for } i \in \{1, \ldots, n\}.$$

5.1.2 Determinants of square matrices

Theorem 5.2. *The functions* $f\colon M_n(\mathbb{F}) \to \mathbb{F}$ *which satsify*

$$\text{if } A, B \in M_n(\mathbb{F}) \quad \text{then} \quad f(AB) = f(A)f(B),$$

are the functions

$$\begin{array}{rccc} \det^k\colon & M_n(\mathbb{F}) & \longrightarrow & \mathbb{F} \\ & A & \longmapsto & \det(A)^k \end{array} \qquad \text{for } k \in \mathbb{Z},$$

where the function $\det\colon M_n(\mathbb{F}) \to \mathbb{F}$ *is determined by*

$$\det(AB) = \det(A)\det(B)$$

and the conditions

$$\det(x_{ij}(c)) = 1, \quad \det(s_i) = -1, \quad \text{and} \qquad \det(h_i(d)) = d,$$

for $i, j \in \{1, \ldots, n\}$ *with* $i < j$, $c \in \mathbb{F}$ *and* $d \in \mathbb{F}$.

5.2 Formulas for determinants

5.2.1 Determinants of permutations

Let $n \in \mathbb{Z}_{>0}$. The *symmetric group* is

$$S_n = \{w\colon \{1, \ldots, n\} \to \{1, \ldots, n\} \mid w \text{ is a bijection}\}.$$

Let $w \in S_n$. The *inversion set of* w is

$$\mathrm{Inv}(w) = \{(i, j) \mid i, j \in \{1, \ldots, n\} \text{ and } i < j \text{ and } w(i) > w(j)\}.$$

Identify w with the $n \times n$ matrix given by $w(i, j) = \begin{cases} 1, & \text{if } j = w(i), \\ 0, & \text{otherwise.} \end{cases}$

Proposition 5.3. *Let* $w \in S_n$. *Then*

$$\det(w) = (-1)^{\ell(w)}, \qquad \text{where} \quad \ell(w) = \#\mathrm{Inv}(w).$$