4 Lecture 4: Level 0 representations

4.1 Extremal weight modules $L(\Lambda)$

Let $\Lambda \in \mathfrak{h}^*_{\mathrm{int}}$. The *extremal weight module* $L(\Lambda)$ is the $\mathbf{U}$-module

$$\begin{aligned} \text{generated by } \{u_{w\Lambda} \mid w \in W\} \qquad &\text{with relations} \quad K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}, \\ E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad &\text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}, \\ F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad &\text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}, \end{aligned} \tag{4.1}$$

for $i \in \{0, \cdots, n\}$. This module has a crystal, denoted $B(\Lambda)$.

4.2 Level 0 extremal weight modules $L(\lambda)$

Let

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n, \qquad \text{with} \quad m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}.$$

Let

$$x_{1,1}, \ldots, x_{m_1,1}, \qquad x_{1,2}, \ldots, x_{m_2,2}, \qquad \ldots, \qquad x_{1,n}, \ldots, x_{m_n,n},$$

be n sets of formal variables and define

$$RG_\lambda = \mathbb{C}[x_{1,1}^{\pm 1}, \ldots, x_{m_1,1}^{\pm 1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{\pm 1}, \ldots, x_{m_n,n}^{\pm 1}]^{S_{m_n}}$$

For $i \in \{1, \ldots, n\}$, define

$$\begin{aligned} e_+^{(i)}(u) &= (1 - x_{1,i}u)(1 - x_{2,i}u) \cdots (1 - x_{m_i,i}u) \quad \text{and} \\ e_-^{(i)}(u^{-1}) &= (1 - x_{1,i}^{-1}u^{-1})(1 - x_{2,i}^{-1}u^{-1}) \cdots (1 - x_{m_i,i}^{-1}u^{-1}). \end{aligned}$$

Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 4.1. *The extremal weight module $L(\lambda)$ is the $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$-module generated by a single vector m_λ with relations*

$$\mathbf{x}_{i,r}^+ m_\lambda = 0, \qquad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda,$$

$$\mathbf{q}_+^{(i)}(u) m_\lambda = K_i \frac{e_+^{(i)}(q^{-1}u)}{e_+^{(i)}(qu)} m_\lambda \qquad \text{and} \qquad \mathbf{q}_-^{(i)}(u^{-1}) m_\lambda = K_i^{-1} \frac{e_-^{(i)}(qu^{-1})}{e_-^{(i)}(q^{-1}u^{-1})} m_\lambda,$$

where $\mathbf{q}_+^{(i)}(u)$ and $\mathbf{q}_-^{(i)}(u^{-1})$ are generating series for loop generators of $\mathbf{U}$.

An alternative presentation of $L(\lambda)$ is as the $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$-module generated by a single vector m_λ with relations

$$\mathbf{x}_{i,r}^+ m_\lambda = 0, \qquad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda,$$

and

$$\mathbf{e}_s^{(i)} m_\lambda = 0 \quad \text{and} \quad \mathbf{e}_{-s}^{(i)} m_\lambda = 0, \qquad \text{for } i \in \{1, \ldots, n\} \text{ and } s \in \mathbb{Z}_{>m_i},$$

4.3 Finite dimensional standard modules $M^{\mathrm{fin}}(a(u))$

A *Drinfeld polynomial* is an n-tuple of polynomials $a(u) = (a^{(1)}(u), \ldots, a^{(n)}(u))$ with $a^{(i)}(u) \in \mathbb{C}[u]$, represented as

$$a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n, \qquad \text{with} \quad a^{(i)}(u) = (u - a_{1,i}) \cdots (u - a_{m_i,i})$$

so that

$$\text{the coefficient of } u^j \text{ in } a^{(i)}(u) \text{ is } \quad e^{(i)}_{m_i - j}(a_{1,i}, \ldots, a_{m_i,i}),$$

the $(m_i - j)$th elementary symmetric function evaluated at the values $a_{1,i}, \ldots, a_{m_i,i}$. Define

$$M^{\mathrm{fin}}(a(u)) = L(\lambda) \otimes_{RG_\lambda} m_{a(u)},$$

where

$$e^{(i)}_k(x_{1,i}, x_{2,i}, \ldots) m_{a(u)} = e^{(i)}_k(a_{1,i}, \ldots, a_{m_i,i}) m_{a(u)}$$

specifies the RG_λ-action on $m_{a(u)}$. In other words, the module $M^{\mathrm{fin}}(a(u))$ is $L(\lambda)$ except that variables $x_{j,i}$ have been specialised to the values $a_{j,i}$.

4.4 Finite dimensional simple modules

Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 4.2. *The standard module*

$$M^{\mathrm{fin}}(a(u)) \quad \textit{has a unique simple quotient} \quad L^{\mathrm{fin}}(a(u))$$

and

$$\begin{array}{ccc} \{\textit{Drinfeld polynomials}\} & \longrightarrow & \{\textit{finite dimensional simple } \mathbf{U}'\textit{-modules}\} \\ a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n & \longmapsto & L^{\mathrm{fin}}(a(u)) \end{array}$$

is a bijection.

4.5 Crystals for level 0 $L(\lambda)$ and $M^{\mathrm{fin}}(a(u))$

Let

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n, \qquad \text{with } m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}.$$

Let $k = \#\{i \in \{1, \ldots, n\} \mid m_i \neq 0\}$ and

$$S^\lambda = \{\vec{\kappa} = (\kappa^{(1)}, \ldots, \kappa^{(n)}) \mid \kappa^{(i)} \text{ is a partition with } \ell(\kappa^{(i)}) < m_i \text{ for } i \in \{1, \ldots, n\}\}.$$

Given λ there are uniquely determined

$$w \in W^{\mathrm{ad}} \text{ and } j \in \mathbb{Z}_{\geq 0} \text{ and } \nu \in A_1 \qquad \text{such that} \qquad w(\nu + \Lambda_0) = -j\delta + \lambda + \Lambda_0.$$

Then the crystal of $L(\lambda)$ is the set

$$B(\lambda) = B(\nu + \Lambda_0)^+_w \times \mathbb{Z}^k \times S^\lambda.$$

and the crystal of $M^{\mathrm{fin}}(a(u))$ is the set

$$B^{\mathrm{fin}}(\lambda) = B(\nu + \Lambda_0)^+_w.$$

4.6 Character formulas

Let

$$0_q = \frac{1}{1-q} + \frac{q^{-1}}{1-q^{-1}} = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots,$$

(although $\frac{q^{-1}}{1-q^{-1}} = \frac{1}{q-1} = \frac{-1}{1-q}$, it is important to note that 0_q is *not* equal to 0, it is a doubly infinite formal series in q and q^{-1}).

Conceptually, the set $\mathbb{Z}^k \times S^\lambda$ is the crystal of RG_λ. Letting $q = e^{-\delta}$, its character is

$$\mathrm{char}(RG_\lambda) = \Big(0_{q^{m_1}} \prod_{k=1}^{m_1-1} \frac{1}{1-q^k}\Big)\Big(0_{q^{m_2}} \prod_{k=1}^{m_2-1} \frac{1}{1-q^k}\Big) \cdots \Big(0_{q^{m_n}} \prod_{k=1}^{m_n-1} \frac{1}{1-q^k}\Big).$$

The character of the crystal $B(\nu+\Lambda_0)^+_w$ is determined by the Demazure character formulas. A pleasant way to express this character is as the evaluation of an electronic Macdonald polynomial,

$$\mathrm{char}(B(\nu+\Lambda_0)^+_w) = E_{w_0\lambda}(q, 0).$$

Putting $\mathrm{char}(RG_\lambda)$ and $\mathrm{char}(B(\nu+\Lambda_0)^+_w)$ together gives

$$\mathrm{char}(B(\lambda)) = \mathrm{char}(B(\nu+\Lambda_0)^+_w)\,\mathrm{char}(RG_\lambda).$$