3 Lecture 3: Extremal weight modules

Let

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{C}\delta + \mathbb{Z}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}.$$

A set of representatives for the W^{ad}-orbits on $\mathfrak{h}^*_{\mathbb{Z}}$ is $(\mathfrak{h}^*)_{\mathrm{int}} = (\mathfrak{h}^*)^+_{\mathrm{int}} \cup (\mathfrak{h}^*)^0_{\mathrm{int}} \cup (\mathfrak{h}^*)^-_{\mathrm{int}}$, where

$$\begin{aligned} (\mathfrak{h}^*)^+_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}, \\ (\mathfrak{h}^*)^0_{\mathrm{int}} &= \mathbb{C}\delta + 0\Lambda_0 + \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\}, \\ (\mathfrak{h}^*)^-_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}. \end{aligned} \tag{3.1}$$

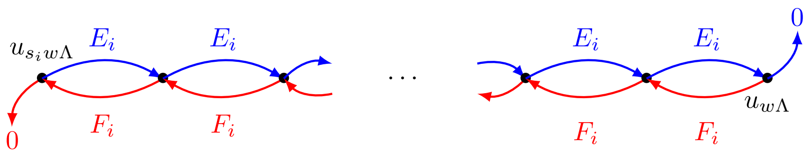
For $\widehat{\mathfrak{sl}}_2$ these sets are pictured (mod δ) in (2.4).

3.1 Extremal weight modules $L(\Lambda)$

Let $\Lambda \in \mathfrak{h}^*_{\mathrm{int}}$. The *extremal weight module* $L(\Lambda)$ is the **U**-module

$$\begin{aligned} \text{generated by } \{u_{w\Lambda} \mid w \in W\} \qquad \text{with relations} \quad & K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}, \\ E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad & \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}, \\ F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad & \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}, \end{aligned} \tag{3.2}$$

for $i \in \{0, \cdots, n\}$. Pictorially, if $\langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ then there is a chain of length $\langle w\Lambda, \alpha_i^\vee\rangle$ from $u_{w\Lambda}$ to $u_{s_i w\Lambda}$,

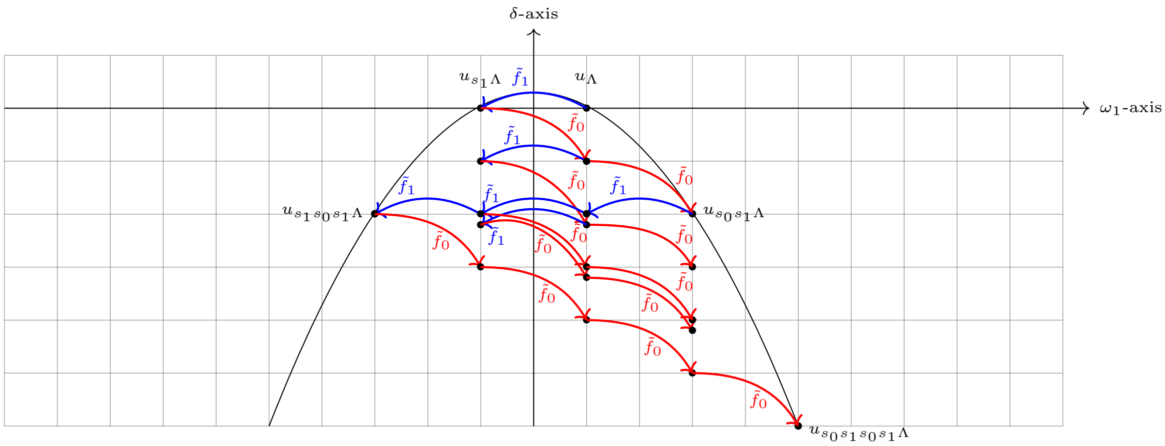


The module $L(\Lambda)$ has a crystal, denoted $B(\Lambda)$. The crystal is a labeling set for a (weight) basis of $L(\Lambda)$.

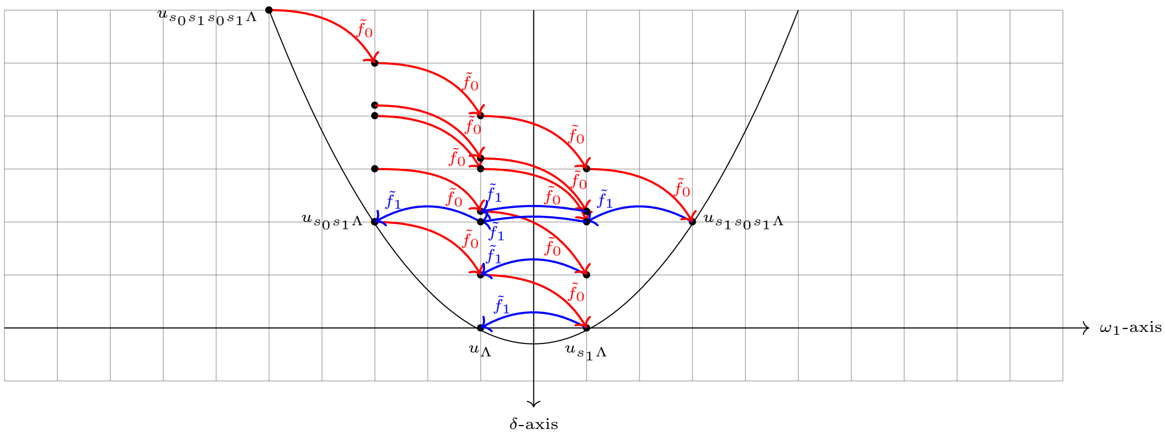
Some properties of the $L(\Lambda)$ are:

- If $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is the simple **U**-module of highest weight Λ.
- If $\Lambda \notin (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is not a highest weight module.
- If $\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$ then $L(\Lambda)$ is the simple **U**-module of lowest weight Λ.
- If $\Lambda \notin (\mathfrak{h}^*)^-_{\mathrm{int}}$ then $L(\Lambda)$ is not a lowest weight module.

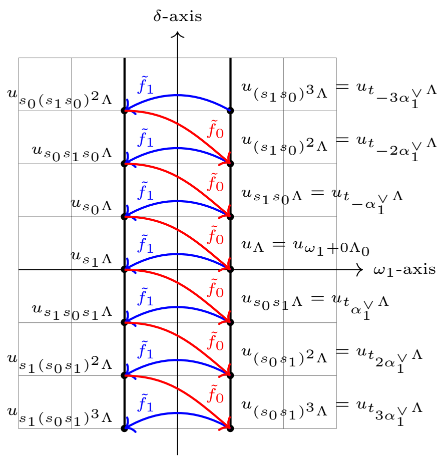
PLATE B: Pictures of $B(\omega_1+\Lambda_0)$, $B(\omega_1+0\Lambda_0)$ and $B(-\omega_1-\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Initial portion of the crystal graph of $B(\omega_1+\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Final portion of the crystal graph of $B(-\omega_1-\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Middle portion of the crystal graph of $B(\omega_1+0\Lambda_0)$ for $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$

3.2 Bruhat orders

Let $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}\text{-span}\{\alpha_1, \ldots, \alpha_n\}$. An *alcove* is a fundamental region for the action of W^{ad} on $(\mathbb{R}\delta + \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0)/\mathbb{R}\delta$. There is a bijection

$$\begin{array}{ccc} W^{\mathrm{ad}} & \longleftrightarrow & \{\text{alcoves}\} \\ 1 & \longmapsto & \{x + \Lambda_0 \in \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0 \mid x(h_i) > 0 \text{ for } i \in \{0, \ldots, n\}\} \end{array} \tag{3.3}$$

An element $w \in W^{\mathrm{ad}}$ is *dominant* if

$$w(\rho + \Lambda_0) \in \mathbb{R}_{\ge 0}\text{-span}\{\omega_1, \ldots, \omega_n\} + \Lambda_0, \qquad \text{where} \quad \rho = \omega_1 + \cdots + \omega_n.$$

In the identification (3.3) of elements of W^{ad} with alcoves, the dominant elements of W^{ad} are the alcoves in the dominant Weyl chamber.

Let $x, w \in W^{\mathrm{ad}}$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w in the generators $s_0, \ldots, s_n$. The *positive level Bruhat order on* W^{ad} is defined by

$$x \overset{+}{\le} w \quad \text{if } x \text{ has a reduced word which is a subword of } w = s_{i_1} \cdots s_{i_\ell}$$

The *negative level Bruhat order* on W^{ad} is defined by $x \le w$ if $x \overset{+}{\ge} w$.

The *level 0 Bruhat order* on W^{ad} is determined by

(a) $\overset{0}{\le}$ for dominant elements: If x, w are dominant then $x \overset{0}{\le} w$ if and only if $x \overset{+}{\le} w$,

(b) $\overset{0}{\le}$ translation invariance: If $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ and $x, w \in W$ then $x \overset{0}{\le} w$ if and only if $xt_{\mu^\vee} \overset{0}{\le} wt_{\mu^\vee}$.

3.3 Demazure submodules

Let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by $E_0, \ldots, E_n, K_0, \ldots, K_n, C, D$.

Let $w \in W^{\mathrm{ad}}$. The *Demazure module* $L(\Lambda)_w^+$ is the $\mathbf{U}^+$-submodule of $L(\Lambda)$ given by

$$L(\Lambda)_w^+ = \mathbf{U}^+ u_{w\Lambda} \qquad \text{and} \qquad \mathrm{char}(L(\Lambda)_w^+) = \sum_{p \in B(\Lambda)_w^+} e^{\mathrm{wt}(p)},$$

since $L(\Lambda)_w^+$ has a crystal $B(\Lambda)_w^+$.

3.4 Demazure operators

The *BGG-Demazure operator* on $\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$ is given by

$$D_i = (1 + s_i) \frac{1}{1 - X^{-\alpha_i}}, \qquad \text{for } i \in \{0, 1, \ldots, n\}.$$

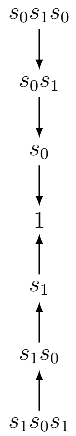
Let $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$, $w \in W^{\mathrm{ad}}$ and $i \in \{0, 1, \ldots, n\}$.

$$\text{If } \Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^+ \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_w^+)) = \begin{cases} \mathrm{char}(L(\Lambda)_{s_i w}^+), & \text{if } s_i w \overset{+}{\ge} w, \\ \mathrm{char}(L(\Lambda)_w^+), & \text{if } s_i w \overset{+}{\le} w; \end{cases}$$

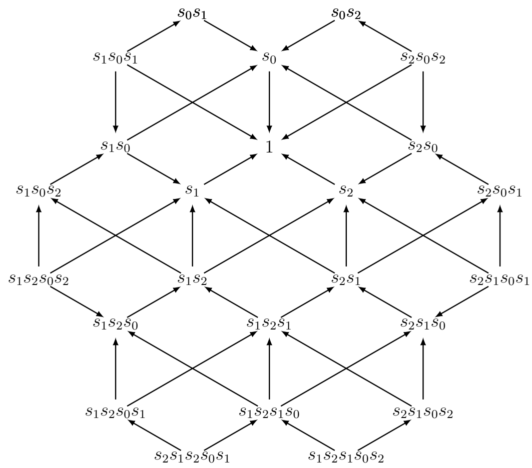
$$\text{if } \lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^0 \quad \text{then} \qquad D_i \mathrm{char}(L(\lambda)_w^+) = \begin{cases} \mathrm{char}(L(\lambda)_{s_i w}^+), & \text{if } s_i w \overset{0}{\ge} w, \\ \mathrm{char}(L(\lambda)_w^+), & \text{if } s_i w \overset{0}{\le} w; \end{cases}$$

$$\text{if } \Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^- \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_w^+) = \begin{cases} \mathrm{char}(L(\Lambda)_{s_i w}^+), & \text{if } s_i w \ge w, \\ \mathrm{char}(L(\Lambda)_w^+), & \text{if } s_i w \le w; \end{cases}$$

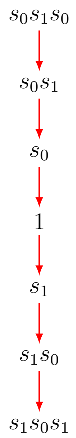
PLATE A: Bruhat orders on the affine Weyl group (partial relations)



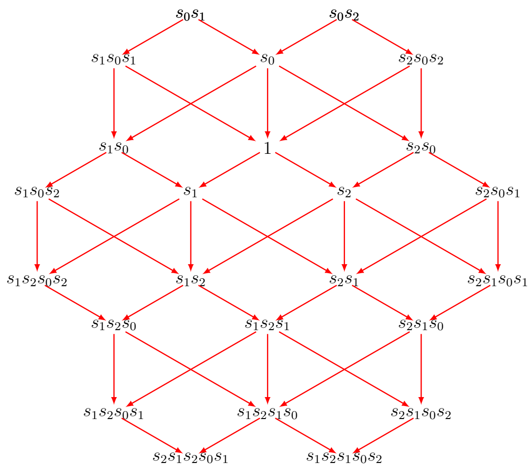
postive level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is minimal



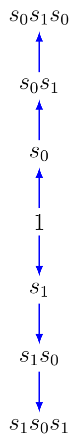
postive level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is minimal



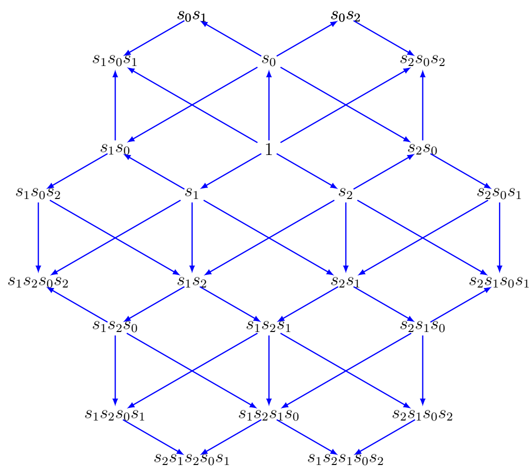
level zero Bruhat order for $\widehat{\mathfrak{sl}}_2$
translation invariant



level zero Bruhat order for $\widehat{\mathfrak{sl}}_3$
translation invariant



negative level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is maximal



negative level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is maximal

3.5 Crystals

Let $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$. The universal crystal in mind is the set

$$B(\text{univ}) = \left\{ \text{piecewise linear paths } p\colon \mathbb{R}_{[0,1]} \to \mathfrak{h}^*_{\mathbb{R}} \;\middle|\; \begin{array}{l} p(0) = 0, \\ p(1) \in \mathfrak{h}^*_{\mathbb{Z}} \end{array} \right\}$$

with root operators $\tilde{e}_0, \ldots, \tilde{e}_n$ and $\tilde{f}_0, \ldots, \tilde{f}_n$ defined by Littelmann (see [Ra06, §5] for an exposition). For $\Lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the *straight line path from* 0 *to* Λ is

$$p_\Lambda\colon \mathbb{R}_{[0,1]} \to \mathfrak{h}^*_{\mathbb{R}} \qquad \text{given by} \qquad p_\Lambda(t) = t\Lambda, \qquad \text{for } t \in \mathbb{R}_{[0,1]}.$$

Let $w \in W^{\text{ad}}$.

$$\text{If } \Lambda \in (\mathfrak{h}^*)^+_{\text{int}} \quad \text{then} \qquad \begin{array}{l} B(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\ge 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}, \\ \\ B(\Lambda)^+_w = \{p \in B(\Lambda) \mid \text{the initial direction of } p \text{ is } \overset{+}{\le} w\}. \end{array}$$

$$\text{If } \Lambda \in (\mathfrak{h}^*)^-_{\text{int}} \quad \text{then} \qquad \begin{array}{l} B(\Lambda) = \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\ge 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}, \\ \\ B(\Lambda)^+_w = \{p \in B(\Lambda) \mid \text{the initial direction of } p \text{ is } \overset{-}{\le} w\}. \end{array}$$

3.6 Weyl character formula

The Weyl character formulas are formulas for the characters of the extremal weight modules $L(\Lambda)$ for the cases when $\Lambda \in (\mathfrak{h}^*)^+_{\text{int}}$ or $\Lambda \in (\mathfrak{h}^*)^-_{\text{int}}$.

Let $\mathring{\rho} = \omega_1 + \cdots + \omega_n$ and $h^\vee = a_0^\vee + a_1^\vee + \cdots + a_n^\vee$ and

$$\rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_n = \omega_1 + \cdots + \omega_n + (a_0^\vee + a_1^\vee + \cdots + a_n^\vee)\Lambda_0 \;= \mathring{\rho} + h^\vee \Lambda_0.$$

Letting

$$q = e^{-\delta},$$

the *Weyl denominators* are

$$a^+_\rho = e^{\mathring{\rho} + h^\vee \Lambda_0} \prod_{r \in \mathbb{Z}_{>0}} \Big((1 - q^r)^n \cdot \prod_{\alpha \in R^+} (1 - q^{r-1} e^{-\alpha})(1 - q^r e^{\alpha}) \Big)$$

and

$$a^-_\rho = e^{-\mathring{\rho} - h^\vee \Lambda_0} \prod_{r \in \mathbb{Z}_{>0}} \Big((1 - q^{-r})^n \cdot \prod_{\alpha \in R^+} (1 - q^{-(r-1)} e^{\alpha})(1 - q^{-r} e^{-\alpha}) \Big)$$

and the *Weyl character formulas* are

$$\text{if } \Lambda \in (\mathfrak{h}^*)^+_{\text{int}} \qquad \text{then} \qquad \text{char}(L(\Lambda)) = \frac{1}{a^+_\rho} \sum_{w \in W} \det(w) e^{w(\Lambda + \rho)},$$

$$\text{if } \Lambda \in (\mathfrak{h}^*)^-_{\text{int}} \qquad \text{then} \qquad \text{char}(L(\Lambda)) = \frac{1}{a^-_\rho} \sum_{w \in W} \det(w) e^{w(\Lambda - \rho)}.$$

The *Weyl denominator formula* is equivalent to $\text{char}(L(0)) = 1$.