2 Lecture 2: The affine Weyl group

2.1 The finite Weyl group W_{fin}

Before we start the game we wish to play we are given some symbols δ, Λ_0 and $\alpha_1^\vee, \ldots, \alpha_n^\vee$ and some integers denoted

$$\begin{matrix} a_0, a_1, \ldots, a_n \in \mathbb{Z}_{>0}, \\ a_0^\vee, a_1^\vee, \ldots, a_n^\vee \in \mathbb{Z}_{>0}, \end{matrix} \qquad \text{and} \qquad C_{ij}, \text{ for } i, j \in \{1, \ldots, n\}.$$

Fix a $\mathbb{C}$-vector space $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ so that

$$\begin{matrix} \mathfrak{a} \text{ has } \mathbb{C}\text{-basis } \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}, \\ \mathfrak{a}^* \text{ has } \mathbb{C}\text{-basis } \{\omega_1, \ldots, \omega_n\} \end{matrix} \qquad \text{with} \qquad \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

Let

$$\mathfrak{h}^* \text{ be the vector space with basis } \quad \{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}.$$

For $j \in \{1, \ldots, n\}$ define

$$\alpha_j = C_{1j}\omega_1 + \cdots + C_{nj}\omega_n$$

and

$$s_j \colon \mathfrak{h}^* \to \mathfrak{h}^* \qquad \text{by} \qquad s_i(a\delta + \lambda + \ell\Lambda_0) = a\delta + \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i + \ell\Lambda_0,$$

for $a, \ell \in \mathbb{C}$ and $\lambda \in \mathfrak{a}^*$. The *finite Weyl group*

$$W_{\mathrm{fin}} \subseteq GL(\mathfrak{h}^*) \qquad \text{is generated by } s_1, \ldots, s_n.$$

2.2 The affine Weyl group

Let

$$\mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}.$$

For $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ define $t_{\mu^\vee} \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$t_{\mu^\vee}(a\delta + \lambda + \ell\Lambda_0) = (a + (\lambda - \ell\tfrac{1}{2}\mu^\vee)(\mu^\vee))\delta + \lambda + \ell\mu^\vee + \ell\Lambda_0,$$

where $a, \ell \in \mathbb{C}$, $\lambda \in \mathfrak{a}^*$ and $\mu^\vee$ is viewed as an element of $\mathfrak{a}^*$ by the isomorphism

$$\begin{matrix} \mathfrak{a} & \xrightarrow{\sim} & \mathfrak{a}^* \\ a_j^\vee \alpha_j^\vee & \mapsto & a_j \alpha_j \end{matrix}$$

The *affine Weyl group* is

$$W^{\mathrm{ad}} = \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} \rtimes W_{\mathrm{fin}} = \{t_{\mu^\vee} w \mid \mu^\vee \in \mathfrak{a}_{\mathbb{Z}},\ w \in W_{\mathrm{fin}}\}$$

with

$$t_{\mu^\vee} t_{\nu^\vee} = t_{\mu^\vee + \nu^\vee} \qquad \text{and} \qquad w t_{\mu^\vee} = t_{w\mu^\vee} w, \qquad \text{for } \mu^\vee, \nu^\vee \in \mathfrak{a}_{\mathbb{Z}} \text{ and } w \in W_{\mathrm{fin}}.$$

2.3 The Heisenberg group

The matrices of s_i and $t_{\mu^\vee}$, with respect to the basis $\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$ of $\mathfrak{h}^*$, are

$$s_i = \left(\begin{array}{c|cccccc|c} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 & -\alpha_i(h_1) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_2) & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_i(h_{i-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_{i+1}) & 1 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_n) & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right), \qquad \text{for } i \in \{1, \ldots, n\}, \text{ and} \tag{2.1}$$

$$t_{\mu^\vee} = \left(\begin{array}{c|ccc|c} 1 & k_1 & \cdots & k_n & -\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle \\ \hline \vdots & & & & \mu_1^\vee \\ 0 & & 1 & & \vdots \\ \vdots & & & & \mu_n^\vee \\ \hline 0 & \cdots & 0 & \cdots & 1 \end{array}\right), \quad \text{for} \quad \begin{aligned} \mu^\vee &= k_1 h_1 + \cdots + k_n h_n \\ &= \tfrac{k_1 a_1}{a_1^\vee}\alpha_1 + \cdots + \tfrac{k_n a_n}{a_n^\vee}\alpha_n \\ &= \mu_1^\vee \omega_1 + \cdots + \mu_n^\vee \omega_n \text{ in } \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}, \end{aligned} \tag{2.2}$$

so that $-\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle = -\frac{1}{2}(\mu_1^\vee k_1 + \cdots \mu_n^\vee k_n)$. The *Heisenberg group* is the subgroup of $GL(\mathfrak{h}^*)$ consisting of transformations for which, with respect to the basis $\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$ of $\mathfrak{h}^*$, the matrices are

$$\left(\begin{array}{c|ccc|c} 1 & a_1 & \cdots & a_n & z \\ \hline \vdots & & & & \gamma_1 \\ 0 & & 1 & & \vdots \\ \vdots & & & & \gamma_n \\ \hline 0 & \cdots & 0 & \cdots & 1 \end{array}\right) \quad \text{with} \quad a_1, \ldots, a_n, \quad \gamma_1, \ldots, \gamma_n, \quad z \text{ in } \mathbb{R}.$$

2.4 Orbit representatives of the W^{ad} action on $(\mathfrak{h}^*)_{\mathrm{int}}$

Let

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{C}\delta + \mathbb{Z}\mathrm{span}\{\omega_1, \ldots, \omega_n, \Lambda_0\}.$$

For $\ell \in \mathbb{Z}_{>0}$, define

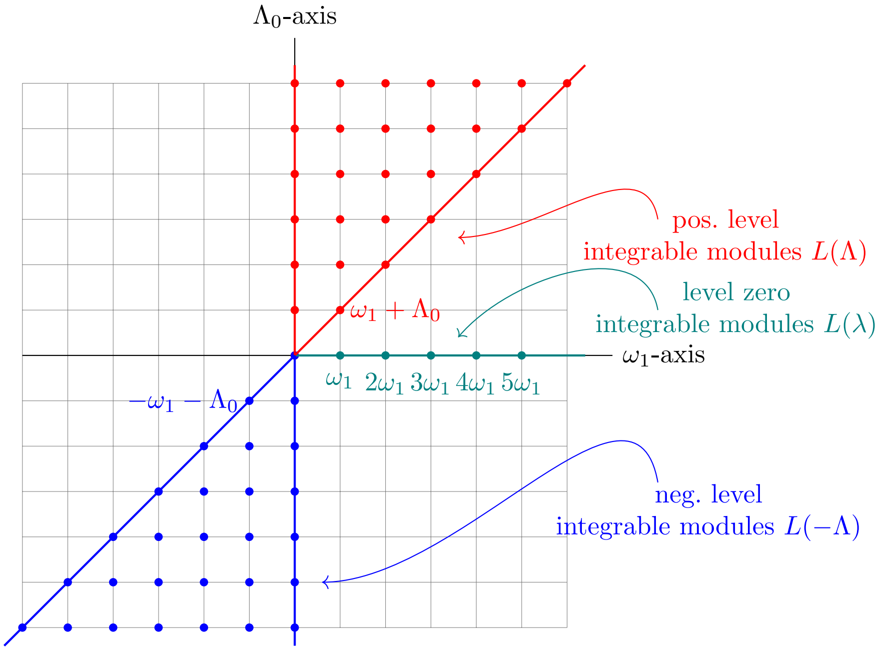
$$A_\ell = \{a\delta + \lambda + \ell\Lambda_0 \in \mathfrak{h}^*_{\mathbb{Z}} \mid a \in \mathbb{C},\ \lambda(h_\theta) \le \ell, \text{ and } \lambda(h_{\alpha_i}) \ge 0 \text{ for } i \in \{1, \ldots, n\}\},$$

$$A_0 = \{a\delta + \lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid a \in \mathbb{C}, \text{ and } \lambda(h_{\alpha_i}) \ge 0 \text{ for } i \in \{1, \ldots, n\}\} \quad \text{and}$$

$$A_{-\ell} = \{a\delta + \lambda + \ell\Lambda_0 \in \mathfrak{h}^*_{\mathbb{Z}} \mid a \in \mathbb{C},\ \lambda(h_\theta) \ge \ell, \text{ and } \lambda(h_{\alpha_i}) \le 0 \text{ for } i \in \{1, \ldots, n\}\}.$$

A set of representatives of the W^{ad} orbits on $\mathfrak{h}^*_{\mathbb{Z}}$ is

$$\mathfrak{h}^*_{\mathrm{int}} = \cdots \cup A_{-2} \cup A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup \cdots.$$



2.5 Classifications of integrable modules

Let $\mathbf{U} = U_t(\mathfrak{g})$ be the quantum group corresponding to the affine Kac-Moody algebra

$$\mathfrak{g} = \mathbb{C}d \oplus \mathring{\mathfrak{g}}[\epsilon, \epsilon^{-1}] \oplus \mathbb{C}K.$$

Let $\mathbf{U}'$ be $\mathbf{U}$ but without the generator $D = t^d$.

Let $\ell \in \mathbb{Z}_{>0}$. The sets A_ℓ provide index sets for classes of $\mathbf{U}$-modules:

$$\begin{aligned}
A_0 &\quad\leftrightarrow\quad \{\text{finite dimensional } U_t(\mathring{\mathfrak{g}}) \text{ modules}\} \\
A_\ell &\quad\leftrightarrow\quad \{\text{level } \ell \text{ irreducible integrable } U_t(\mathfrak{g}) \text{ modules}\} \\
A_{-\ell} &\quad\leftrightarrow\quad \{\text{level } -\ell \text{ irreducible integrable } U_t(\mathfrak{g}) \text{ modules}\} \\
A_0 &\quad\leftrightarrow\quad \{\text{level 0 integrable extremal weight modules } U_t(\mathfrak{g}) \text{ modules}\}
\end{aligned}$$

Let

$$\mathbb{C}[u]^{\oplus n}_{\mathrm{mon}} = \{a_1(u)\omega_1 + \cdots + a_n(u)\omega_n \mid a_i(u) \in \mathbb{C}[u] \text{ and } a_i(u) \text{ is monic}\}$$

Then

$$\mathbb{C}[u]^{\oplus n}_{\mathrm{mon}} \leftrightarrow \{\text{irreducible finite dimensional } U_t(\mathfrak{g})\text{-modules}\}$$

2.6 Type $A_1^{(1)}$

For Type $A_1^{(1)}$ the initial data consists of the matrices $B = DC$ given by

$$B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = DC.$$

Since

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{then} \qquad K = \alpha_0^\vee + \alpha_1^\vee \quad \text{and} \quad \Lambda_1 = \omega_1 + \Lambda_0.$$

The matrices

$$t_{k\alpha_1^\vee} = \begin{pmatrix} 1 & -k & -k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } k \in \mathbb{Z},$$

generate the action of W^{ad} on $\mathfrak{h}^*$, with respect to the basis $\{\delta, \omega_1, \Lambda_0\}$ of $\mathfrak{h}^*$, For example, if $\ell \in \mathbb{R}_{\neq 0}$ then

$$\text{the } W^{\mathrm{ad}}\text{-orbit of } (2\omega_1 + \ell\Lambda_0) \text{ is contained in the parabola} \quad \{y\delta + x\omega_1 + \ell\Lambda_0 \mid y = \tfrac{1}{-4\ell}(x^2 - 4)\}.$$

and if $\ell = 0$ then

$$\text{the } W^{\mathrm{ad}}\text{-orbit of } (2\omega_1 + 0\Lambda_0) \text{ is contained in the two lines} \quad \{y\delta + x\omega_1 \mid x = 2 \text{ or } x = -2\}.$$

These parabolas and lines are displayed in the following picture.