1 Lecture 1: Examples of U-modules

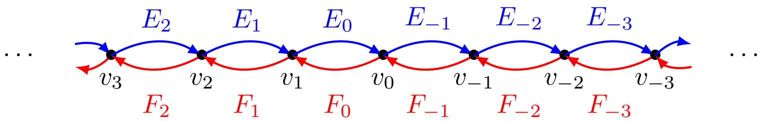
1.1 The module $L(\varepsilon_1)$ for $\mathbf{U} = U_t(\mathfrak{sl}_\infty)$

Let $E_{ij} \in M_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i,j) entry and 0 elsewhere.
Let

$$L(\varepsilon_1) \quad \text{be a vector space with basis} \quad \{v_i \mid i \in \mathbb{Z}\}.$$

For $i \in \mathbb{Z}$, define

$$E_i = E_{i,i+1}, \qquad F_i = E_{i+1,i}, \qquad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}.$$



1.2 The module $L(\varepsilon_1)$ for $\mathbf{U} = U_t(L\mathfrak{sl}_n)$

Let $n \in \mathbb{Z}_{>2}$.
Let $E_{ij} \in M_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i,j) entry and 0 elsewhere.
Let

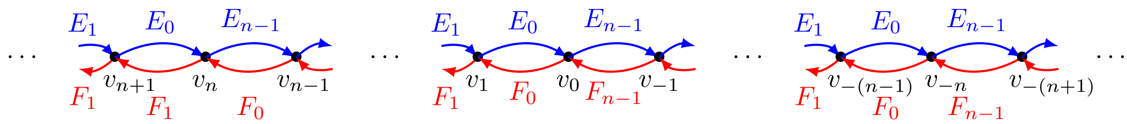
$$L(\varepsilon_1) \quad \text{be a vector space with basis} \quad \{v_i \mid i \in \mathbb{Z}\}.$$

For $i \in \mathbb{Z}/n\mathbb{Z}$, define

$$E_i = \sum_{\substack{k\in\mathbb{Z}\\ k=i \bmod n}} E_{k,k+1}, \qquad F_i = \sum_{\substack{k\in\mathbb{Z}\\ k=i \bmod n}} E_{k+1,k},$$

and

$$K_i = 1 + \sum_{\substack{k\in\mathbb{Z}\\ k=i \bmod n}} (t-1)E_{kk} + (t^{-1}-1)E_{k+1,k+1}.$$



1.3 Coiling $L(\varepsilon_1)$ for $U_t(L\mathfrak{sl}_n)$

Let $n \in \mathbb{Z}_{>2}$.
Let $E_{ij} \in M_{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}(\mathbb{C}[\epsilon,\epsilon^{-1}])$ denote the matrix with 1 in the (i,j) entry and 0 elsewhere.
Let $\{v_0, v_1, \ldots, v_{n-1}\}$ be a basis of $\mathbb{C}^n$. Then the vector space

$$\mathbb{C}^n[\epsilon,\epsilon^{-1}] = \mathbb{C}[\epsilon,\epsilon^{-1}] \otimes_{\mathbb{C}} \mathbb{C}^n \qquad \text{has } \mathbb{C}\text{-basis} \qquad \{\epsilon^\ell v_i \mid i \in \mathbb{Z}/n\mathbb{Z}, \ell \in \mathbb{Z}\}.$$

Define $\mathbb{C}[\epsilon,\epsilon^{-1}]$-linear endomorphisms of $\mathbb{C}^n[\epsilon,\epsilon^{-1}]$ by

$$E_i = E_{i,i+1}, \qquad F_i = E_{i+1,i}, \qquad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}, \qquad \text{for } i \in \{1,\ldots,n-1\},$$

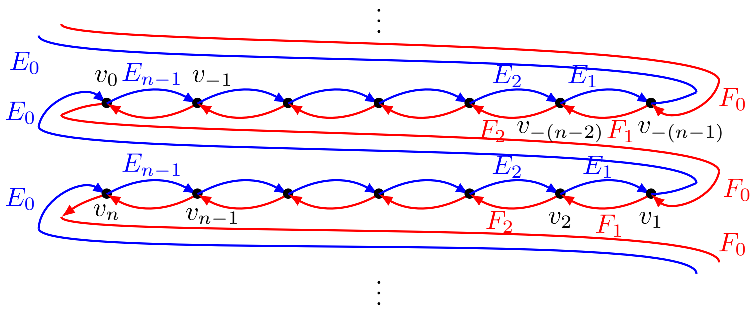
and define E_0, F_0, K_0 by

$$E_0 = \epsilon E_{0,1}, \qquad F_0 = \epsilon^{-1}E_{1,0}, \qquad K_0 = 1 + (t-1)E_{0,0} + (t^{-1}-1)E_{1,1},$$

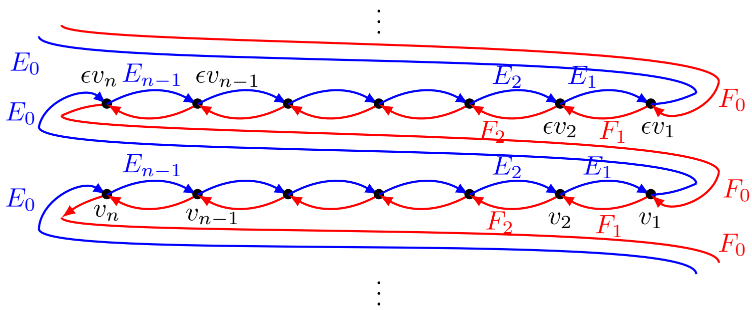
Then

$$\begin{array}{ccc} \mathbb{C}^n[\epsilon,\epsilon^{-1}] & \to & L(\varepsilon_1) \\ \epsilon^\ell v_i & \mapsto & v_{i-\ell n} \end{array} \qquad \text{is an isomorphism of } U_t(L\mathfrak{sl}_n)\text{-modules.}$$

Pictorially,



is isomorphic to



1.4 The module $L^{\mathrm{fin}}(u_1 - a)$ for $U_t(L\mathfrak{sl}_n)$

Let $a \in \mathbb{C}$.
Let $n \in \mathbb{Z}_{>1}$ and let $E_{ij} \in M_{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i,j) entry and 0 elsewhere. Let

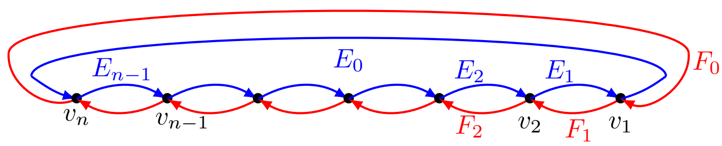
$$\mathbb{C}^n \qquad \text{be a vector space with basis} \qquad \{v_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}.$$

For $i \in \{1,\dots,n\}$, define

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}, \qquad \text{for } i \in \{1,\dots,n-1\},$$

and define E_0, F_0, K_0 by

$$E_0 = aE_{0,1}, \qquad F_0 = a^{-1}E_{1,0}, \qquad K_0 = 1 + (t-1)E_{0,0} + (t^{-1}-1)E_{1,1},$$



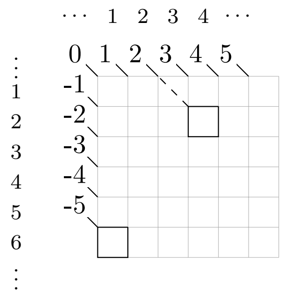
HW: Assume that t is not a root of unity. Prove that $L^{\mathrm{fin}}(u_1 - a)$ is an irreducible $U_t(L\mathfrak{sl}_n)$-module.
HW: Assume that t is not a root of unity. Prove that if $a_1, a_2 \in \mathbb{C}$ with $a_1 \neq a_2$ then

$$L^{\mathrm{fin}}(u_1 - a_1) \qquad \text{is not isomorphic to} \qquad L^{\mathrm{fin}}(u_1 - a_2).$$

1.5 Skew shapes and column strict tableaux

A *box* is an element of $\mathbb{Z}^2$ (rows and columns are indexed as for matrices). The *content* of box $= (i, j)$ is the diagonal number of the box (i, j),

$$c(\text{box}) = j - i.$$

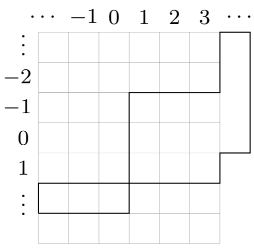


$b = (2, 4)$ has content $c(b) = 2$,
$b = (6, 1)$ has content $c(b) = -5$.

A *skew shape* is a finite subset of $\mathbb{Z}^2$ such that

If $r \in \mathbb{Z}_{>0}$ and $(i, j), (i + r, j + r) \in \nu/\mu$ then $(i + a, j + b) \in \nu/\mu$ for $a, b \in \{0, 1, \ldots, r\}$.

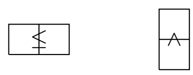
$\nu/\mu =$



$$= \left\{ \begin{array}{l} (-3,4), (-2,4), \\ (-1,1), (-1,2), (-1,3), (-1,4), \\ (0,1), (0,2), (0,3), (0,4), \\ (1,1), (1,2), (1,3), \\ (2,-2), (2,-1), (2,0), \end{array} \right\}$$

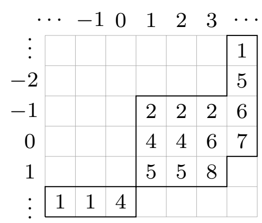
A *column strict tableau of shape ν/μ filled from* $\{1, \ldots, n\}$ is a function $T \colon \nu/\mu \to \{1, \ldots, n\}$ such that

(a) if $(i, j), (i + 1, j) \in \nu/\mu$ then $T(i, j) > T(i + 1, j)$,
(b) if $(i, j), (i, j + 1) \in \nu/\mu$ then $T(i, j) \leq T(i, j + 1)$.



For example,

$T =$



is a column strict tableau filled from $\{1, 2, \ldots, 9\}$.

Let

$$B(\nu/\mu) = \{\text{column strict tableaux of shape } \nu/\mu \text{ filled from } \{1, \ldots, n\}\}.$$

The set $B(\nu/\mu)$ is empty if ν/μ contains a column of length $> n$. If $B(\nu/\mu)$ is nonempty then it contains he *column reading tableau* $T^+ \in B(\nu/\mu)$ determined by

(a) if $(i, j) \in \nu/\mu$ and $(i - 1, j)) \in \nu/\mu$ then $T^+(i, j) = 1$,
(b) if $(i, j), (i, j + 1) \in \nu/\mu$ then $T^+(i, j + 1) = T^+(i, j) + 1$.

1.6 The $\mathbf{U}'$-module $L^{\mathrm{fin}}(\nu/\mu)$

Let $\mathbf{U}' = U_t(L\mathfrak{sl}_n)$ and let

$$L^{\mathrm{fin}}(\nu/\mu) \qquad \text{be the vector space with basis} \qquad \{v_T \mid T \in B(\nu/\mu)\}.$$

For $T \in B(\nu/\mu)$, define

$$\gamma_T(u_0, u_1, \ldots, u_{n-1}, u_n) = \prod_{b\in\nu/\mu} \frac{1 - u_{T(b)}t^{2c(b)+T(b)-1}}{1 - u_{T(b)-1}t^{2c(b)+T(b)}}.$$

For $i \in \{1, \ldots, n-1\}$, let $\gamma_T^{(i)}(u)$ be $\gamma_T(u_0, u_1, \ldots, u_{n-1}, u_n)$ evaluated at $u_i = u$ and $u_j = 0$ at $j \neq i$,

$$\gamma_T^{(i)}(u) = \gamma_T(0, \ldots, 0, u, 0, \ldots, 0).$$

For $i \in \{1, \ldots, n\}$ and $u, w, z \in \mathbb{C}^\times$, define operators $\mathbf{q}^{(i)}(u)$, $\mathbf{x}_i^+(t^a)$ and $\mathbf{x}_i^-(t^a)$ on $L^{\mathrm{fin}}(\nu/\mu)$ by

$$\mathbf{q}_+^{(i)}(u)v_T = q^{\deg(\gamma_T^{(i)})}\frac{\gamma_T^{(i)}(t^{-1}u)}{\gamma_T^{(i)}(tu)}v_T, \qquad \text{and} \qquad \mathbf{q}_-^{(i)}(u)v_T = q^{-\deg(\gamma_T^{(i)})}\frac{\gamma_T^{(i)}(tu)}{\gamma_T^{(i)}(t^{-1}u)}v_T,$$

and

$$\mathbf{x}_i^+(t^a)v_T = \begin{cases} (const)v_{\tilde{e}_{i,a}T}, & \text{if } T \text{ has a box } b \text{ with } T(b) = i \text{ and } c(b) = a, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{x}_i^-(t^a)v_T = \begin{cases} (const)v_{\tilde{f}_{i,a}T}, & \text{if } T \text{ has a box } b \text{ with } T(b) = i+1 \text{ and } c(b) = a, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\tilde{e}_{i,a}T \text{ is } T \text{ except with } i \text{ changed to } i+1 \text{ in a box of content } a,$$

$$\tilde{f}_{i,a}T \text{ is } T \text{ except with } i+1 \text{ changed to } i \text{ in a box of content } a,$$