16 Vector spaces with topology

16.1 Topological vector spaces

Let $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ with $i^2 = -1$ be the field of complex numbers with complex conjugation

$$\begin{matrix} \mathbb{C} & \to & \mathbb{C} \\ c & \mapsto & \overline{c} \end{matrix} \qquad \text{given by} \qquad \overline{a+bi} = a - bi,$$

and absolute value

$$\begin{matrix} \mathbb{C} & \to & \mathbb{R}_{\geq 0} \\ c & \mapsto & |c| \end{matrix} \qquad \text{given by} \qquad |c|^2 = c\,\overline{c}.$$

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-*vector space* is a set V with functions

$$\begin{matrix} V \times V & \to & V \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{K} \times V & \to & V \\ (c, v) & \mapsto & cv \end{matrix}$$

(*addition* and *scalar multiplication*) such that

(a) If $v_1, v_2, v_3 \in V$ then $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$,

(b) There exists $0 \in V$ such that if $v \in V$ then $0 + v = v$ and $v + 0 = v$,

(c) If $v \in V$ then there exists $-v \in V$ such that $v + (-v) = 0$ and $(-v) + v = 0$,

(d) If $v_1, v_2 \in V$ then $v_1 + v_2 = v_2 + v_1$,

(e) If $c \in \mathbb{K}$ and $v_1, v_2 \in V$ then $c(v_1 + v_2) = cv_1 + cv_2$,

(f) If $c_1, c_2 \in \mathbb{K}$ and $v \in V$ then $(c_1 + c_2)v = c_1 v + c_2 v$,

(g) If $c_1, c_2 \in \mathbb{K}$ and $v \in V$ then $c_1(c_2 v) = (c_1 c_2)v$,

(h) If $v \in V$ then $1v = v$.

A *topological field* is a field $\mathbb{K}$ with a topology such that

$$\begin{matrix} \mathbb{K} \times \mathbb{K} & \to & \mathbb{K} \\ (a, b) & \mapsto & a + b \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{K} \times \mathbb{K} & \to & \mathbb{K} \\ (a, b) & \mapsto & ab \end{matrix} \qquad \text{are continuous.}$$

Let $\mathbb{K}$ be a topological field. A *topological* $\mathbb{K}$-*vector space* is a $\mathbb{K}$-vector space V with a topology such that

$$\begin{matrix} V \times V & \to & V \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{K} \times V & \to & V \\ (c, v) & \mapsto & cv \end{matrix} \qquad \text{are continuous.}$$

16.1.1 Normed vector spaces and Banach spaces

A *normed vector space* is a $\mathbb{K}$-vector space V with a function $\| \ \| \colon V \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$,

(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\,\|v\|$,

(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

Let $(V, \| \ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d \colon V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

A *Banach space* is a normed vector space V which is complete (as a metric space with the norm metric).

16.1.2 Inner product spaces and Hilbert spaces

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

A *positive definite symmetric inner product space* is a $\mathbb{K}$-vector space V with a function

$$\begin{matrix} V \times V & \to & \mathbb{K} \\ (v_1, v_2) & \mapsto & \langle v_1, v_2\rangle \end{matrix} \qquad \text{such that}$$

(a) (symmetry condition) If $v_1, v_2 \in V$ then $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$,

(b) (linearity in the first coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle c_1v_1 + c_2v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle$,

(c) (linearity in the second coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle v_3, c_1v_1 + c_2v_2\rangle = c_1\langle v_3, v_1\rangle + c_2\langle v_3, v_2\rangle$,

(d) (diagonal condition) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

(e) (norm condition) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$.

A *positive definite Hermitian inner product space* is a $\mathbb{K}$-vector space V with a function

$$\begin{matrix} V \times V & \to & \mathbb{K} \\ (v_1, v_2) & \mapsto & \langle v_1, v_2\rangle \end{matrix} \qquad \text{such that}$$

(a) (symmetry condition) If $v_1, v_2 \in V$ then $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$,

(b) (linearity in the first coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle c_1v_1 + c_2v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle$,

(c) (conjugate linearity in the second coordinate) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2, v_3 \in V$ then $\langle v_3, c_1v_1 + c_2v_2\rangle = \overline{c_1}\langle v_3, v_1\rangle + \overline{c_2}\langle v_3, v_2\rangle$,

(d) (diagonal condition) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

(e) (norm condition) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$.

An *inner product space* is a positive definite symmetric inner product space or a positive definite Hermitian inner product space.

Let $(V, \langle,\rangle)$ be an inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

A *Hilbert space* is an inner product space V which is complete (as a metric space with the norm metric for the length norm).

Theorem 16.1. *Let $(V, \langle,\rangle)$ be an inner product space.*

(a) (Pythagorean theorem) If $x, y \in V$ and $\langle x, y\rangle = 0$ then $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.

(b) (Parallelogram law) If $x, y \in V$ $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

(c) (Cauchy-Schwarz) If $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

(d) (triangle inequality) If $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$.

16.2 Bounded linear operators

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed $\mathbb{K}$-vector spaces. The space of *bounded linear operators from V to W* is

$$B(V, W) = \{T : V \to W \mid T \text{ is linear and } \|T\| \text{ exists in } \mathbb{R}_{\geq 0}\},$$

where

$$\|T\| = \sup\Big\{\frac{\|Tx\|_W}{\|x\|_V} \ \Big|\ x \in H\Big\}.$$

Proposition 16.2. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed $\mathbb{K}$-vector spaces. Let $T : V \to W$ be a linear operator. The following are equivalent.*

(a) T is bounded.

(b) T is continuous.

(c) T is uniformly continuous.

16.2.1 Duals and adjoints

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a normed $\mathbb{K}$-vector space. The space of *bounded linear functionals on V*, or the *dual of V*, is

$$V^* = B(V, \mathbb{K}) = \{\text{bounded linear operators } \varphi : V \to \mathbb{K}\}.$$

Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed vector spaces. Let $T : V \to W$ be a linear operator. The *adjoint of T* is the linear transformation

$$T^* : W^* \to V^* \quad \text{given by} \quad (T^*\varphi)(v) = \varphi(T(v)).$$

Proposition 16.3. *Let H be a Hilbert space. Then*

$$\begin{array}{rccc} \Psi: & H & \longrightarrow & H^* \\ & x & \longmapsto & \Psi_x \end{array} \quad \text{where} \quad \begin{array}{rccc} \Psi_x: & H & \to & \mathbb{K} \\ & h & \to & \langle h, x\rangle \end{array}$$

is a skew-linear bijective isometry and $\|\Psi\| = 1$.

The dual H^* does not have a natural inner product so it is not naturally a Hilbert space until it is identified with H. The proof of Proposition 24.1 uses Theorem 17.2.

16.3 Bases

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let V be a $\mathbb{K}$-vector space.
A *basis of V* is a subset $B \subseteq V$ such that

(a) $\mathbb{K}\text{-span}(B) = V$,

(b) B is linearly independent,

where

$$\mathbb{K}\text{-span}(B) = \{a_1 b_1 + \cdots + a_\ell b_\ell \mid \ell \in \mathbb{Z}_{>0},\ b_1, \ldots, b_\ell \in B,\ a_1, \ldots, a_\ell \in \mathbb{K}\}$$

and B is *linearly independent* if B satisfies

$$\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } b_1, \ldots, b_\ell \in B \text{ and } a_1, \ldots, a_\ell \in \mathbb{K}, \text{ and}$$

$$a_1 b_1 + \cdots a_\ell b_\ell = 0 \quad \text{then} \quad a_1 = 0,\ a_2 = 0,\ \ldots,\ a_\ell = 0.$$

Let V be a topological $\mathbb{K}$-vector space. A *topological basis of V* is a subset $B \subseteq V$ such that

(a) $\overline{\mathbb{K}\text{-span}(B)} = V$,

(b) B is linearly independent,

Proposition 16.4. *Let $\mathbb{K}$ be $\mathbb{R}$ of $\mathbb{C}$ and let $(V, \| \ \|)$ be a normed $\mathbb{K}$-vector space. Then V has a countable dense set C if and only if V has a sequence $B = (b_1, b_2, \ldots)$ with $\overline{\mathbb{K}\text{-span}(B)} = V$.*

Let V be a topological $\mathbb{K}$-vector space. A *Schauder basis of* V is a sequence $(b_1, b_2, \ldots)$ in V such that

$$\text{if } v \in V \text{ then there exists a unique sequence } (a_1, a_2, \ldots) \in \mathbb{K} \text{ such that } \quad \sum_{i \in \mathbb{Z}_{>0}} a_i b_i = v,$$

where $v = \sum_{i \in \mathbb{Z}_{>0}} a_i b_i$ means

$$v = \lim_{n \to \infty} s_n \quad \text{where} \quad s_1 = a_1 b_1, \ s_2 = a_1 b_1 + a_2 b_2, \ s_3 = a_1 b_1 + a_2 b_2 + a_3 b_3, \ \ldots.$$

16.4 Orthogonality

16.4.1 Orthonormal sequences and Gram-Schmidt

Let V be a Hilbert space.
An *orthonormal sequence* in V is a sequence $(a_1, a_2, \ldots)$ in V such that

$$\text{if } i, j \in \mathbb{Z}_{>0} \quad \text{then} \qquad \langle a_i, a_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Proposition 16.5. *Let V be an inner product space.*

(a) An orthonormal sequence $(a_1, a_2, \ldots)$ in V is linearly independent.

(b) (Gram-Schmidt) Let $(v_1, v_2, \ldots)$ be a sequence of linearly independent vectors in V. Let

$$a_1 = \frac{v_1}{\|v_1\|}, \qquad \text{and} \qquad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n\|}.$$

Then $(a_1, a_2, \ldots)$ is an orthonormal sequence of linearly independent vectors in V.

Theorem 16.6. *Let H be a Hilbert space. If H has a countable dense set then $H \cong \ell^2$.*

16.4.2 Orthogonals and projections in Hilbert spaces

Let V be a inner product space and let $S \subseteq V$. The *orthogonal* to S is

$$S^\perp = \{v \in V \mid \text{if } w \in S \text{ then } \langle v, w \rangle = 0\}.$$

Let $x \in V$. The *distance from x to S* is

$$d(x, S) = \inf\{d(x, w) \mid w \in W\}.$$

Proposition 16.7. *Let H be a Hilbert space and let W be a closed subspace of H.*

(a) If $x \in H$ then there exists a unique $y \in W$ such that $d(x, y) = d(x, W)$.

(b) *Define $P: H \to H$ by setting $P(x) = y$ where y is as in (a). Then P is a linear transformation,*

$$P(x) \in W, \qquad (\mathrm{id} - P)(x) \in W^\perp, \qquad \|P\| = 1,$$

$$P^2 = P, \qquad (\mathrm{id} - P)^2 = (\mathrm{id} - P), \qquad \textit{and} \qquad \mathrm{id} = P + (\mathrm{id} - P).$$

Let H be a Hilbert space, let W be a closed subspace. The *projection onto* W is the bounded linear transformation $P_W: H \to H$ given by

$$P_W(x) = y, \quad \text{where} \quad y \in W \text{ is such that } d(x, y) = d(x, W).$$

Theorem 16.8. *Let V be a Hilbert space. Let W be a subset of V.*

(a) *$W^\perp$ is a closed subspace of V.*

(b) *W is a closed subspace of V if and only if $V = W \oplus W^\perp$.*

Theorem 16.9. *Let H be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H, let*

$$W = \mathbb{K}\text{-}span\{a_1, a_2, \ldots\}, \quad \overline{W} \textit{ the closure of } W, \qquad \textit{and} \quad P_{\overline{W}}: H \to H$$

the projection onto $\overline{W}$. If $x \in H$ then

$$P_{\overline{W}}(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n,$$

16.5 Eigenvectors and eigenspaces

16.5.1 Eigenvalues and compact operators

Let H be a complex vector space and let $T: H \to H$ be a linear operator. Let $\lambda \in \mathbb{C}$. The λ-*eigenspace of* T is

$$X_\lambda = \{v \in H \mid Tv = \lambda v\} = \ker(T - \lambda) \qquad \text{and} \qquad \sigma_p(T) = \{\lambda \in \mathbb{C} \mid X_\lambda \neq 0\}.$$

is the *point spectrum of* T.

Let X be a normed vector space and let $T: X \to X$ be a bounded linear operator.

- T is *compact* if T satisfies:

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } \{x \in H \mid \|x\| = 1\}$$
$$\text{then } (Tx_1, Tx_2, \ldots) \text{ has a cluster point in } X.$$

Proposition 16.10. *Let H be a Hilbert space and let $\lambda \in \mathbb{C}$.*

(a) *Let $T: H \to H$ be a linear operator. Then*

$$T \textit{ has an eigenvector of eigenvalue } \lambda \quad \textit{if and only if} \quad \lambda - T \textit{ is not injective.}$$

(b) *(Fredholm's theorem) Let $T: H \to H$ be a compact linear operator. Then*

$$\lambda - T \textit{ is injective} \qquad \textit{if and only if} \qquad \lambda - T \textit{ is bijective.}$$

16.5.2 Existence of eigenvectors

Let H be a Hilbert space and let $T: H \to H$ be a bounded linear operator.

- T is *self adjoint* if T satsifies: if $x, y \in H$ then $\langle Tx, y\rangle = \langle x, Ty\rangle$.
- T is an *isometry* if T satisfies: if $x, y \in H$ then $\langle Tx, Ty\rangle = \langle x, y\rangle$.
- T is *unitary* if T is an isometry and T is invertible.

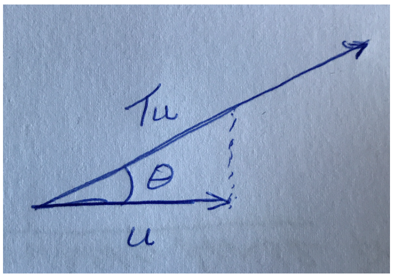
If $T: H \to H$ is a self adjoint operator and $u \in H$ then

$$\langle Tu, u\rangle = \langle u, Tu\rangle = \overline{\langle Tu, u\rangle} \quad \text{so that} \quad \langle Tu, u\rangle \in \mathbb{R}.$$

The Cauchy-Schwarz inequality gives

$$|\langle Tu, u\rangle| \le \|Tu\| \cdot \|u\| \qquad \text{and} \qquad \theta = \cos^{-1}\left(\frac{\langle Tu, u\rangle}{\|Tu\| \cdot \|u\|}\right)$$

is the "angle between Tu and u" . If $\theta = 0$ or $\theta = \pi$ then there exists $\lambda \in \mathbb{C}$ such that $Tu = \lambda u$ and u is an eigenvector of T. The angle θ will be 0 or π when $\|\langle Tu, u\rangle\|$ acheives the maximum possible value $\|Tu\| \cdot \|u\|$. This intuition is made precise by the following two theorems.



Theorem 16.11. *Let H be a Hilbert space and let $T: H \to H$ be a bounded self adjoint linear operator. Let*

$$\lambda = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}.$$

Then

$$\lambda = \|T\| \qquad \textit{and} \qquad \lambda - T \textit{ is not a bijection.}$$

Theorem 16.12. *Let H be a Hilbert space and let $T: H \to H$ be a compact self adjoint linear operator. Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u_n\| = 1\}$ such that*

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \|T\| \qquad \textit{and let } y \textit{ be a cluster point of } Tu_1, Tu_2, \ldots.$$

Then

$$\|y\| = \|T\|, \qquad \frac{|\langle Ty, y\rangle|}{\|y\|^2} = \|T\| \qquad \textit{and} \qquad Ty = \|T\|\, y.$$

Let H be a Hilbert space and let $b_0 \in H$. The *Rayleigh quotient* is

$$\mu_{k+1} = \frac{\langle Ab_k, b_k\rangle}{\langle b_k, b_k\rangle} = \frac{\langle b_{k+1}, b_k\rangle}{\|b_k\|^2}, \qquad \text{where} \quad b_{k+1} = \frac{Ab_k}{\|Ab_k\|} = \frac{A^{k+1}b_0}{\|A^{k+1}b_0\|}. \tag{16.1}$$

Theorem 16.13. *Let H be a Hilbert space, let $b_0 \in H$ and let $(b_1, b_2, \ldots)$ and $(\mu_1, \mu_2, \ldots)$ be defined by (18.1). Then*

$$\lim_{k\to\infty} b_{k+1} = b \quad \textit{is an eigenvector} \quad \textit{of eigenvalue} \quad \lambda_1 = \lim_{k\to\infty} \mu_k.$$

16.5.3 Eigenspace decomposition and the spectral theorem

Let H be a vector space and let $T: H \to H$ be a linear operator. Let $\lambda \in \mathbb{K}$. The *λ-eigenspace of T* is

$$H_\lambda = \{v \in H \mid Tv = \lambda v\}. \qquad \text{and} \qquad \sigma_p(T) = \{\lambda \in \mathbb{K} \mid H_\lambda \neq 0\}$$

is the *point spectrum of T*.

Proposition 16.14. *Let $T: H \to H$ be a self adjoint operator. For $\lambda \in \mathbb{K}$ let H_λ be the λ-eigenspace of T.*

(a) If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

(b) If $\lambda \neq \gamma$ then $H_\lambda \perp H_\gamma$.

(c) If T is compact and $\lambda \neq 0$ then H_λ is finite dimensional.

(c) If is compact and $\lambda_1, \lambda_2, \ldots$ is a sequence of distinct eigenvalues of T then

$$\lim_{n\to\infty} \lambda_n = 0.$$

Theorem 16.15. *(Hilbert-Schmidt spectral theorem) Let H be a Hilbert space. Let $T: H \to H$ be a bounded compact self adjoint linear operator.*

(a) Then

$$H = \overline{W}, \qquad \text{where} \quad W = \bigoplus_{\lambda\in\sigma_p(T)} H_\lambda.$$

(b) If H has a countable dense set then there exists a countable orthonormal basis of eigenvectors of T.

16.6 Notes and references

Proposition 24.1 is often called the "Reisz representation theorem" (see [Bre, Theorem 5.7]), but should not be confused with another similar theorem which is also often called the Reisz representation theorem (see [Ru, Theorem 2.14]). An alternative source for the initial results of this chapter, including the results on orthogonality and the Reisz representation theorem, is [Bre, Ch. 5].

The Hilbert-Schmidt theorem, Theorem 18.6, establishes that compact self adjoint operators are diagonalizable. Proposition 18.5 provides an outline for the proof. The crucial step that compact self adjoint operators have an eigenvector with eigenvalue equal to the norm is the content of Theorem 18.3 . An alternative reference for these results and Fredholm's theorem is [Bre, Ch. 6]. References for power iteration and the Rayleigh quotient are [Wil] and [TB]
(see also https://en.wikipedia.org/wiki/List_of_numerical_analysis_topics#Eigenvalue_algorithms).

In functional analysis nonseparable Hilbert spaces (Hilbert spaces which do not have a countable dense set) are relatively rare (see mathoverflow and other resources for examples).

There are four kinds of conditions:

(a) bilinearity

(b) symmetry conditions: symmetric, skew-symmetric, unitary

(c) isotropy conditions

(d) positive definiteness

The purpose of a condition like $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$ is to make sure that $\|\ \|\colon V \to \mathbb{R}_{\geq 0}$ defined by

$$\|v\|^2 = \langle v, v\rangle \quad \text{has image in } \mathbb{R}_{\geq 0} \quad \text{to give us a norm.}$$

Many people (Pete Clark) take norms to have values in $\mathbb{R}_{\geq 0}$ and valuations (logs of norms) to be in a totally ordered abelian group (Atiyah-Macdonald).

The motivation for the discovery of the Baire category theorem and the corresponding results about bounded linear operators was from the attempt to try to extend the derivative map from a subspace where it is defined to the whole HIlbert space. GET A GOOD REFERENCE/SUMMARY.

THERE ARE GOOD FORMULAS FOR THE SECOND LARGEST ETC EIGENVALUES AS SUP $\langle Tu, u\rangle|$ FOR u RUNNING OVER 2-DIMENSIONAL SUBSPACES.